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2008 J. Phys. A: Math. Theor. 41 115203

(http://iopscience.iop.org/1751-8121/41/11/115203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 115203 (28pp) doi:10.1088/1751-8113/41/11/115203

Characterizing singular curves in parametrized families of biquadratics

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Received 8 November 2007, in final form 27 November 2007
Published 4 March 2008
Online at stacks.iop.org/JPhysA/41/115203

Abstract
We consider families of biquadratic curves $B = 0$ on $\mathbb{C}^2$, defined with respect to arbitrarily many complex parameters. Due to the fact that these families include curve intersections across different parameter combinations, they represent a generalization of the non-intersecting foliations of one-parameter invariant curves associated with the QRT mapping. We use algebraic methods involving discriminants to provide a complete classification of the singular curves in these families. In developing this classification, we exploit the special symmetric nature of B; namely, that it is a quadratic in x and y whose reflection in the line $y = x$ is given by a simple change of parameters. We also define a range of conditions in the biquadratic's parameters and demonstrate the manner in which they correspond to different geometric realizations of the singular curves.

PACS numbers: 02.30.Ik, 02.40.Xx, 02.40.−k

Introduction

A family of *biquadratic* curves in the complex (x, y)-plane is described by

$$\begin{aligned} B(x, y; K_1, \ldots, K_q) = {} & \alpha(K_1, \ldots, K_q)x^2y^2 + \beta(K_1, \ldots, K_q)x^2y + \delta(K_1, \ldots, K_q)xy^2 \\ & + \gamma(K_1, \ldots, K_q)x^2 + \kappa(K_1, \ldots, K_q)y^2 + \epsilon(K_1, \ldots, K_q)xy \\ & + \xi(K_1, \ldots, K_q)x + \lambda(K_1, \ldots, K_q)y + \mu(K_1, \ldots, K_q) = 0, \end{aligned} \tag{1}$$

where each of the coefficients $\alpha, \ldots, \mu$ is an expression in the parameters $K_1, \ldots, K_q \in \mathbb{C}$, with $q \geqslant 1$. The study of biquadratic curves has a long history, dating back to e.g. Euler (and his number theoretic results involving biquadratic reciprocity) and Frobenius (see [4]). In the discrete integrable dynamics context, particular cases of biquadratic curve families appear in [5] and originally in [10, 11], being preserved in the latter case by integrable birational mappings of the plane of the form

$$L: \quad \mathbb{C}^2 \to \mathbb{C}^2: \quad x' = \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)}, \qquad y' = \frac{g_1(x') - yg_2(x')}{g_2(x') - yg_3(x')}. \tag{2}$$

The f_i and g_i are certain quartic polynomials whose coefficients are functions of $\alpha, \ldots, \mu$ in (1) (see [5, 10, 11] for an explicit representation of these functions). Typically, the biquadratic (1) is an elliptic curve and the action of L is equivalent to translation on the associated Weierstrass curve (see [3, 6][1] for recent work and further references).

Here, however, we are concerned not so much with the dynamics on (1) but on the nature of the curve itself. In particular, in contrast to [6], we are interested in the case where (1) is not elliptic owing to its possession of at least one singular point in the affine plane (see definition 1). We want to know how to find these *singular curves* in a parameterized family, the number and location of singular points lying on any given singular curve and, more broadly, the nature of the geometry of these curves. Importantly, while integrable maps involve families (1) whose curves are nested and non-intersecting, we want to be able to ask these questions irrespective of any such foliation condition.

In order to introduce two explicit examples that we meet again throughout the paper, let $\mathbf{K} = \{K_2, \ldots, K_q\}$, and distinguish the first parameter by $K_1 = t$ (in the case where $q = 2$, we denote $\mathbf{K}$ simply by K). We shall call

$$\begin{aligned} B_{\mathrm{QRT}}(x, y; t, \mathbf{K}) = {} & \big(\alpha_{Q_1}(\mathbf{K})t + \alpha_{Q_0}(\mathbf{K})\big)x^2y^2 + \big(\beta_{Q_1}(\mathbf{K})t + \beta_{Q_0}(\mathbf{K})\big)x^2y \\ & + \big(\delta_{Q_1}(\mathbf{K})t + \delta_{Q_0}(\mathbf{K})\big)xy^2 + \big(\gamma_{Q_1}(\mathbf{K})t + \gamma_{Q_0}(\mathbf{K})\big)x^2 \\ & + \big(\kappa_{Q_1}(\mathbf{K})t + \kappa_{Q_0}(\mathbf{K})\big)y^2 + \big(\epsilon_{Q_1}(\mathbf{K})t + \epsilon_{Q_0}(\mathbf{K})\big)xy \\ & + \big(\xi_{Q_1}(\mathbf{K})t + \xi_{Q_0}(\mathbf{K})\big)x + \big(\lambda_{Q_1}(\mathbf{K})t + \lambda_{Q_0}(\mathbf{K})\big)y + \mu_{Q_1}(\mathbf{K})t + \mu_{Q_0}(\mathbf{K}) \\ = {} & 0, \end{aligned} \tag{3}$$

which possesses coefficients that are affine in t, the *QRT biquadratic*, and

$$\begin{aligned} B_M(x, y; t, \mathbf{K}) = {} & \alpha_M(\mathbf{K})x^2y^2 + \beta_M(\mathbf{K})x^2y + \delta_M(\mathbf{K})xy^2 + \gamma_M(\mathbf{K})x^2 + \kappa_M(\mathbf{K})y^2 \\ & + \epsilon_M(\mathbf{K})xy + \xi_M(\mathbf{K})x + \lambda_M(\mathbf{K})y + \mu_M(\mathbf{K}) - t = 0, \end{aligned} \tag{4}$$

whose t-parameter appears only in the constant coefficient, the *McMillan biquadratic* (after [7]). When $q = 2$, we have chosen for expository reasons to specify t as the representative of these biquadratics' level set heights.

This paper closely examines the qualitative behaviour of curve families such as (1), (3) and (4) under parameter variation, focussing on parameter combinations for which singular curves emerge. We shall see that by working with a particular set of parameter constraints, these curves can be classified according to the nature of the singular points they possess (specifically, by the multiplicity of their x- and y-coordinates when represented as the roots of certain discriminant functions). In the general case, the simultaneous satisfaction of (up to six) such parameter constraints defines a hypersurface in some subset of $\{K_1, \ldots, K_q\}$, from which one can smoothly select parameter combinations associated with a particular singularity class[2].

Before commencing with the theoretical exposition, let us consider by way of motivation the McMillan biquadratic

$$\begin{aligned} B_{\mathrm{M}} = {} & 2x^2y^2 + (3K+6)x^2 + (3K+6)y^2 - \left(\tfrac{283}{500}K^2 + \tfrac{2897}{500}K + \tfrac{1577}{250}\right)xy \\ & - 8x - \tfrac{1}{5}y + K + 1 - t = 0. \end{aligned} \tag{5}$$

[1] We thank Professor Duistermaat for sending us a copy of his manuscript.

[2] We note that in the algebraic geometric approach [3] to the study of the QRT map acting on (3), considered as a rational elliptic surface, it is crucial to first identify the type of the singular curves in the curve family.

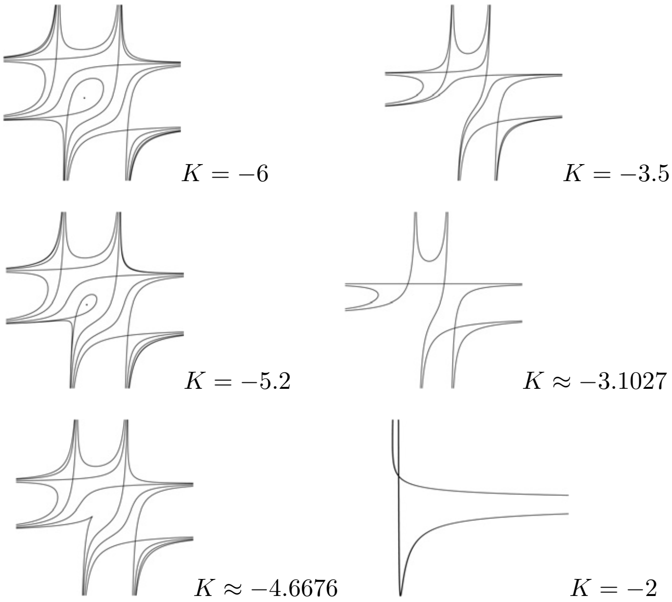


Figure 1. Families of biquadratic curves (5) parameterized by t for six different values of K.

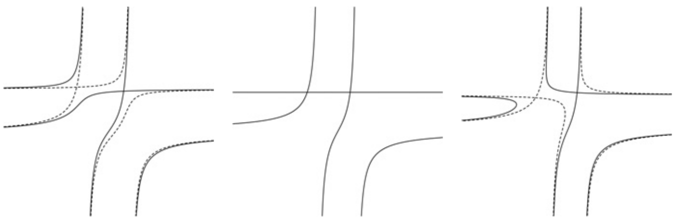


Figure 2. A sequence of plots illustrating the bifurcation of singular curves of (5). At $(K, t) = (K^*, t^*) \approx (-3.1027, -7.8318)$, (5) possesses two singularities, $P_1 \approx (-2.0393, 1.2861)$ and $P_2 \approx (0.8302, 1.2861)$, with the same y-coordinate (middle plot). Associated with each K near K^* are two singular curves of (5), each with one singular point. They approach one another as K approaches K^* from below (represented by the dashed and undashed curves in the left plot). After merging at $K = K^*$, the two singular curves dissociate again as K increases beyond K^* (right plot).

As K is varied, the topology of each curve family represented by (5) changes. Figure 1 includes a sample of six such families, where only the real singular level sets are shown.

The sequence of plots on the left-hand side of the figure highlights the fact that at $K \approx -4.6676$, two singular level sets merge at a singular level set characterized by a cusp[3]. The same is true at $K \approx -3.1027$, except that here the emergent singular level set possesses an entire horizontal line. A more detailed picture of the latter case is given in figure 2.

Using ideas developed in the ensuing two sections, it will be shown that at any parameter combination (K, t) satisfying $f_{2X} = 240\,267K^5 + 539\,9640K^4 + 403\,715\,31K^3 + 115\,889\,466K^2 + 139\,488\,804K + 916\,862\,96 = 0$ and $\mu = \mu_{\mathrm{M}} - t = \mu_{d,x} = (240\,267K^5 + 552\,6990K^4 + 691\,845\,81K^3 + 285\,032\,766K^2 + 474\,380\,604K + 313\,363\,496)/(600\,0000(K + 2))$, (5) possesses a singular curve of the type represented in the middle plot of figure 2 and

[3] Defined to be a point on the curve whose associated cuspidal tangent intersects the curve in a triple root.

case 2 of table 2[4]. It will also be shown that any such parameter combination lies on the 0-contour of the surface given by $\mathrm{discr}_{yx}(B)$ (see definition 2), which in a sense made clear by proposition 1 encodes the singular curve families of the biquadratic. An explicit instance of such a combination is given in figure 3, which is a real planar plot of $\mathrm{discr}_{yx}(B)$ for $B = B_{\mathrm{M}}$ of (5). The pair $(K, t) \approx (-3.1027, -7.8318)$, corresponding to the singular curve in the middle plot of figure 2, is identified as the point f.

The plan of this paper is as follows. In section 1, we define the singular points under review. We introduce various discriminants and iterated discriminants of B to establish an alternative method for computing these points and analysing the qualitative nature of the curves to which they belong. This leads to a partial classification (table 1) of the singular curves of (1). We also highlight the important role played by the iterated discriminant $\mathrm{discr}_{yx}(B)$ (referred to above) in characterizing these singular curves.

Section 2 includes a range of results validating the necessary and sufficient conditions listed in table 2, which provide a complete classification of the singular curves of (1) according to their geometry and the configuration of the discriminant factorizations of (35) (and which, therefore, extend the classification of table 1).

1. The singular curves of $B = 0$ and related discriminants

Due to the centrality of their role in this investigation, we begin by defining the singular points of (1). These points constitute an affine variety whose defining equations, $B = \frac{\partial B}{\partial x} = \frac{\partial B}{\partial y} = 0$, are well known. We shall see that the same variety can be defined using the system $B = \mathrm{discr}_x(B) = \mathrm{discr}_y(B) = 0$, whose latter elements (discriminants of B, defined presently) each consist of one less variable than B. When it comes to computing the biquadratic's singularities, this system provides a useful, simpler, alternative to the conventional system.

Remark 1. We stress here that we are concerned with the singular points of (1) in the affine plane. It is easily shown that, working projectively, $B(x, y) = 0$ of (1) becomes a homogeneous triquadratic $B_p(X, Y, Z) = 0$, where $x = X/Z$ and $y = Y/Z$. This projective curve possesses two singular points at infinity: $[X_1, Y_1, Z_1] = [1, 0, 0]$ and $[X_2, Y_2, Z_2] = [0, 1, 0]$.

It will be convenient in what follows to express B as a quadratic in either x or y:

$$\begin{aligned} B(x, y) &= \alpha_2 y^2 + \alpha_1 y + \alpha_0 \\ &= \beta_2 x^2 + \beta_1 x + \beta_0 = 0, \end{aligned} \tag{6}$$

where

$$\begin{array}{lll} \alpha_2 = \alpha x^2 + \delta x + \kappa, & \alpha_1 = \beta x^2 + \epsilon x + \lambda, & \alpha_0 = \gamma x^2 + \xi x + \mu \\ \beta_2 = \alpha y^2 + \beta y + \gamma, & \beta_1 = \delta y^2 + \epsilon y + \xi, & \beta_0 = \kappa y^2 + \lambda y + \mu. \end{array} \tag{7}$$

We adopt the notational convention that $\alpha_2 \equiv 0$ if and only if each of the parameters, α, δ and κ, in α_2 vanishes (similarly for β_2).

We shall denote the discriminant of B with respect to y (respectively x) by $\mathrm{discr}_y(B)$ ($\mathrm{discr}_x(B)$). We have

$$\begin{aligned} \mathrm{discr}_y(B) &= \alpha_1^2 - 4\alpha_0\alpha_2 \\ &= \mathrm{discr}_y(\beta_2)x^4 + (2\beta\epsilon - 4\alpha\xi - 4\delta\gamma)x^3 + (2\beta\lambda + \epsilon^2 - 4(\delta\xi + \gamma\kappa + \alpha\mu))x^2 \\ &\quad + (2\epsilon\lambda - 4\kappa\xi - 4\delta\mu)x + \mathrm{discr}_y(\beta_0) \end{aligned} \tag{8}$$

[4] Provided $\alpha\epsilon - \beta\delta = (-283K^2 - 2897K - 3154)/250 \neq 0$ and $\mathrm{discr}_x(P_{2X}) = (-208\,650\,24K^4 - 914\,441\,0112K^3 - 542\,545\,182\,72K^2 - 107\,952\,832\,512K - 690\,875\,924\,48)/625 \neq 0$.

and

$$\begin{aligned}\mathrm{discr}_x(B) &= \beta_1^2 - 4\beta_0\beta_2\\ &= \mathrm{discr}_x(\alpha_2)y^4 + (2\delta\epsilon - 4\alpha\lambda - 4\beta\kappa)y^3 + (2\delta\xi + \epsilon^2 - 4(\beta\lambda + \gamma\kappa + \alpha\mu))y^2\\ &\quad + (2\epsilon\xi - 4\gamma\lambda - 4\beta\mu)y + \mathrm{discr}_x(\alpha_0).\end{aligned} \tag{9}$$

Putting these definitions on a more general footing, note that for any polynomial

$$F(z) = c_n z^n + \cdots + c_0 \in \mathbb{C}[z] \tag{10}$$

$(n > 1, c_n \neq 0)$ with roots $r_1, \ldots, r_n$,

$$\mathrm{discr}_z(F) = c_n^{2n-2} \prod_{\substack{i,j\\ i<j}}^{n} (r_i - r_j)^2 = \frac{(-1)^{n(n-1)/2}}{c_n} \mathrm{res}_z\left(F, \frac{\partial F}{\partial Z}\right), \tag{11}$$

where $\mathrm{res}_z\left(F, \frac{\partial F}{\partial z}\right)$—the resultant of F and $\frac{\partial F}{\partial z}$ with respect to z—is typically calculated using the Sylvester matrix (see [2, p77] and (17) for an example). Note that when $n = 1$, $\mathrm{discr}_z(F) := 1$. Throughout what follows, we make use of the following important result.

Lemma 1. *There exist polynomials $U(c_0, \ldots, c_n)$, $V(c_0, \ldots, c_n)$ such that for F in (10),*

$$\mathrm{discr}_z(F) = \frac{(-1)^{n(n-1)/2}}{c_n}\left(U\frac{\partial F}{\partial z} + VF\right). \tag{12}$$

Proof. See (11) and theorem 1.3.2 of [9]. □

Definition 1. *The singular points of (1) comprise the variety*

$$B_S = \left\{P \in \mathbb{C}^2 \times \mathbb{C}^q \,\middle|\, B(P) = \frac{\partial B}{\partial x}(P) = \frac{\partial B}{\partial y}(P) = 0\right\}. \tag{13}$$

If for some fixed parameter combination $K_1, \ldots, K_q$, (1) possesses at least one element of B_S, the entire set of points satisfying (1) shall be referred to as a singular curve. If the biquadratic is either QRT or McMillan, we shall use the equivalent description singular level set (defined by some height t for fixed **K***).*

Defining

$$B_S^* = \{P \in \mathbb{C}^2 \times \mathbb{C}^q \,|\, B(P) = \mathrm{discr}_y(B)(P) = \mathrm{discr}_x(B)(P) = 0\}, \tag{14}$$

we have

Lemma 2. *If $\alpha_2 \not\equiv 0$ and $\beta_2 \not\equiv 0$, then $B_S = B_S^*$.*

Proof. The fact that for any quadratic $Q = c_2z^2 + c_1z + c_0$,

$$\mathrm{discr}_z(Q) = \left(\frac{\partial Q}{\partial z}\right)^2 - 4c_2Q \tag{15}$$

ensures that if $P \in B_S$ then $B(P) = \frac{\partial B}{\partial x}(P) = \frac{\partial B}{\partial y}(P) = 0$ and so $\mathrm{discr}_y(B)(P) = \mathrm{discr}_x(B)(P) = 0$. The converse is shown by a similar argument. □

Note that if $\alpha_2 \equiv 0$, then $\mathrm{discr}_y(B) = 1$ and clearly $B_S^* = \emptyset$. But $B_S \neq \emptyset$ in this case as for $B = K_1x^2y^2 + K_2x^2y + K_3xy^2 + K_4x^2 + K_5y^2 + K_6xy + K_7x + K_8y + K_9 = x^2y + x^2 + xy + x + y + 1$, we have $((-1 + \sqrt{3})/2, -1; 0, 1, 0, 1, 0, 1, 1, 1, 1) \in B_S$.

Also note that the presence of the squared term on the right-hand side of (15), which ensures $\nabla B(P) = (0, 0)$ when B, $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ are zero, is special. If F

is a non-biquadratic polynomial in x and y with coefficients in $\mathbb{C}$, only the weaker claim that

$$P : F(P) = \frac{\partial F}{\partial x}(P) = \frac{\partial F}{\partial y}(P) = 0 \Rightarrow F(P) = \mathrm{discr}_y(B)(P) = \mathrm{discr}_x(B)(P) = 0 \tag{16}$$

can be made. This is because lemma 1 ensures that each of $\mathrm{discr}_y(F)$ and $\mathrm{discr}_x(F)$ is expressible as a linear combination of F and its partial derivatives.

To see why the reverse implication does not apply here, consider $F = y((x-1)^2y^2 + x)$. This polynomial, as well as $\mathrm{discr}_y(F) = -4(x-1)^2x^3$ and $\mathrm{discr}_x(F) = -y^2(2y-1)(2y+1)$, vanishes at $(x, y) = (1, 0)$. But $\frac{\partial F}{\partial y}(1, 0) = 3y^2x^2 - 6y^2x + 3y^2 + x|_{(x,y)=(1,0)} = 1$. The 0-contour of F in this case is a 'horizontal fork' with three left-oriented tines meeting at the origin. The central tine is the horizontal axis itself.

Definition 2. *We shall denote the discriminant of* $\mathrm{discr}_y(B)$ *(respectively* $\mathrm{discr}_x(B)$*) with respect to* x (y) *by* $\mathrm{discr}_{yx}(B)$ $(\mathrm{discr}_{xy}(B))$. *It is easily verified that* $\mathrm{discr}_{yx}(B) = \mathrm{discr}_{xy}(B)$[5], *meaning that the two discriminants can be referred to interchangeably.*

This so-called 'double discriminant' (which consists of 1010 terms in $\alpha, \ldots, \mu$ and is a quintic in μ) plays a special role in characterizing the singular curves of (1). In particular, it helps us to determine the maximum number of singular points contained within any given singular curve and to more easily locate these points. In the QRT and McMillan cases, it also allows us to specify a bound on the number of singular level sets possessed by either of these biquadratics for a fixed combination of parameters $\mathbf{K}$.

Lemma 3. *Assume* $\alpha_2, \beta_2 \not\equiv 0$. *Then there exist polynomials* $F_1, F_2, F_3 \in \mathbb{C}[x, y]$ *with coefficients in* $K_1, \ldots, K_q$ *for which* $\mathrm{discr}_{yx}(B) = F_1B + F_2\frac{\partial B}{\partial x} + F_3\frac{\partial B}{\partial y}$.

Proof. The proof is constructive. For the quartic $F(z) = c_4z^4 + \cdots + c_0$, the polynomials

$$\begin{aligned}
f_1 = {} & 32c_4{}^2c_3c_2c_0z - 112c_4{}^2c_3c_2c_1z^2 - 80c_4c_3{}^2c_2c_1z + 16c_4c_2{}^4 - 4c_3{}^2c_2{}^3 + 256c_4{}^3c_0{}^2 \\
& - 6c_3{}^3c_2{}^2z + 96c_4{}^2c_2c_1{}^2 + 18c_3{}^4c_1z + c_4c_3{}^2c_1{}^2 + 32c_4{}^2c_2{}^3z^2 - 128c_4{}^2c_2{}^2c_0 \\
& + 144c_4{}^3c_1{}^2z^2 + 15c_2c_3{}^3c_1 - 27c_3{}^4c_0 + 24c_4c_3c_2{}^3z - 8c_4c_3{}^2c_2{}^2z^2 + 24c_4c_3{}^3c_1z^2 \\
& - 16c_4{}^2c_2{}^2c_1z + 120c_4{}^2c_3c_1{}^2z + 48c_4{}^2c_3{}^2c_0z^2 - 68c_4c_2{}^2c_3c_1 - 192c_4{}^3c_1c_0z \\
& + 144c_2c_3{}^2c_4c_0 - 128c_4{}^3c_2c_0z^2 - 176c_4{}^2c_0c_3c_1, \\
f_2 = {} & 18c_4c_1{}^2c_2c_3 - c_4c_3{}^2c_1{}^2z - 6c_4c_3{}^3c_1z^3 - 39c_4{}^2c_3c_1{}^2z^2 + 4c_4{}^2c_2{}^2c_1z^2 - 42c_4{}^2c_2c_1{}^2z \\
& - 7c_3{}^3c_2c_1z - 12c_2{}^2c_3c_4c_0 + 48c_4{}^3c_1c_0z^2 + 28c_4{}^2c_3c_2c_1z^3 + 27c_4c_3{}^2c_2c_1z^2 \\
& + 32c_4c_3c_2{}^2c_1z - 50c_4c_3{}^2c_2c_0z - 7c_3{}^2c_4c_0c_1 + 48c_1c_4{}^2c_2c_0 - 12c_4{}^2c_3{}^2c_0z^3 \\
& + 56c_4{}^2c_3c_1c_0z - 3c_4c_3{}^3c_0z^2 + 32c_4{}^3c_2c_0z^3 + 48c_4{}^2c_2{}^2c_0z - 8c_4c_2{}^4z - 8c_4{}^2c_2{}^3z^3 \\
& + 2c_3{}^2c_2{}^3z - 16c_3c_4{}^2c_0{}^2 - 64c_4{}^3c_0{}^2z + c_1c_2{}^2c_3{}^2 - 4c_4c_2{}^3c_1 - 6c_3{}^4c_1z^2 - 36c_4{}^3c_1{}^2z^3 \\
& + 2c_3{}^3c_2{}^2z^2 + 3c_3{}^3c_2c_0 + 9c_3{}^4c_0z - 4c_3{}^3c_1{}^2 - 27c_4{}^2c_1{}^3 - 8c_4c_3c_2{}^3z^2 + 2c_4c_3{}^2c_2{}^2z^3
\end{aligned} \tag{17}$$

ensure that

$$\mathrm{discr}_z(F) = f_1F + f_2\frac{\partial F}{\partial z}. \tag{18}$$

Taking $F(x) = \mathrm{discr}_y(B)$ and substituting each of the five coefficients in (8) for their correspondents $c_4, \ldots, c_0$ in (17), we have polynomials $\bar{f}_1$ and $\bar{f}_2$ for which

$$\mathrm{discr}_{yx}(B) = \bar{f}_1\,\mathrm{discr}_y(B) + \bar{f}_2\frac{\partial}{\partial x}(\mathrm{discr}_y(B)). \tag{19}$$

[5] This equality is also shown in [3, 4].

The fact that both $\mathrm{discr}_y(B)$ (care of (15)) and

$$\frac{\partial}{\partial x}\mathrm{discr}_y(B) = 2\frac{\partial^2 B}{\partial x\partial y}\frac{\partial B}{\partial y} - 4\frac{\partial \alpha_2}{\partial x}B - 4\alpha_2\frac{\partial B}{\partial x} \tag{20}$$

can be expressed as linear combinations of B, $\frac{\partial B}{\partial x}$ and $\frac{\partial B}{\partial y}$, gives

$$\mathrm{discr}_{yx}(B) = F_1 B + F_2\frac{\partial B}{\partial x} + F_3\frac{\partial B}{\partial y}, \tag{21}$$

where

$$\begin{aligned} F_1 &= -4\left(\bar{f}_1\alpha_2 + \bar{f}_2\frac{\partial \alpha_2}{\partial x}\right),\\ F_2 &= -4\bar{f}_2\alpha_2,\\ F_3 &= \bar{f}_1\frac{\partial B}{\partial y} + 2\bar{f}_2\frac{\partial^2 B}{\partial x\partial y}. \end{aligned} \tag{22}$$

□

For reference, the discriminant of the quartic $F(z) = c_4 z^4 \cdots + c_0$ used above is given by

$$\begin{aligned} \mathrm{discr}_z(F) = {} & c_1^2c_2^2c_3^2 - 4c_1^3c_3^3 - 4c_1^2c_2^3c_4 + 18c_1^3c_2c_3c_4 - 27c_1^4c_4^2 + 256c_0^3c_4^3 \\ & + c_0\left(-4c_2^3c_3^2 + 18c_1c_2c_3^3 + 16c_2^4c_4 - 80c_1c_2^2c_3c_4 - 6c_1^2c_3^2c_4 + 144c_1^2c_2c_4^2\right) \\ & + c_0^2\left(-27c_3^4 + 144c_2c_3^2c_4 - 128c_2^2c_4^2 - 192c_1c_3c_4^2\right). \end{aligned} \tag{23}$$

Example 1. For the polynomial $B_{\mathrm{QRT}} = x^2y + Ky^2x + x^2 + tx + K$, formulae (22) look like

$$\begin{aligned} F_1 = {} & -256K^4(-16t^2K^3 + 72x^3K^2 + 12x^2t^2K + 64x^2t^2K^2 + 64K^4 + 72tK^3 + 27K^3 \\ & - 54txK^2 - 128txK^3 - 144t^2xK^2 - 192x^2K^3 + 84tx^3K + 32t^3xK^2 - 16t^3K^2 \\ & - 81x^2K^2 - 212x^2tK^2 + 27x^3K - 24t^3x^3 + 64x^2t^3K - 24t^2x^3K + 32t^4xK), \\ F_2 = {} & 256K^4(32t^3xK^2 + 32x^2t^3K + 32t^4xK - 8t^3x^3 + 32x^2t^2K^2 - 8t^2x^3K - 64K^4 \\ & + 4xK^3 - 39x^2K^2 + 24x^3K^2 + 9x^3K - 96x^2K^3 + 16t^3K^2 - 72tK^3 + 16t^2K^3 \\ & - 108x^2tK^2 + 28tx^3K - 112txK^3 - 128t^2xK^2 + 4x^2t^2K - 42txK^2 - 27K^3)x, \\ F_3 = {} & -256K^4(-24x^3K^2y + 8t^2x^3K - 28tx^3K - 27yK^3 - 64yK^4 - 27xK^2 - 64xK^3 \\ & + 3x^2K^2 + 21K^2x^2y - 72K^3ty + 16t^3K^2y + 16t^2K^3y + 16x^2t^4 + 52tK^2x^2y \\ & - 16t^3Kx^2y - 16t^2K^2x^2y - 9yx^3K + 2yxK^3 - 52x^2tK^2 - 60x^2t^2K - 18x^2tK \\ & + 8t^2x^3Ky + 16x^2t^3K + 16t^3xK + 8t^3x^3y + 16t^2xK^2 + 8t^3x^3 - 28tx^3Ky + 8t^2xK^2y \\ & + 48K^3x^2y + 8yxtK^3 - 9x^3K + 6tK^2xy - 24x^3K^2 - 72txK^2 - 4t^2Kx^2y). \end{aligned}$$

For the remainder of this paper, we adopt the following.

Assumption 1. The discriminants $\mathrm{discr}_x(\alpha_2)$ and $\mathrm{discr}_y(\beta_2)$ (see (7)) are non-zero.

A clear consequence of this assumption is that $\alpha_2 \not\equiv 0$, $\beta_2 \not\equiv 0$ (as, for example, $\alpha_2 \equiv 0 \Leftrightarrow \alpha = \delta = \kappa = 0 \Rightarrow \mathrm{discr}_x(\alpha_2) = 0$).

Proposition 1.

(a) If $P \in B_S$, then $\mathrm{discr}_{yx}(B)(P) = 0$ *and*

- *(i) the parameter combination $K_1, \ldots, K_q$ for the singular curve containing P lies on the 0-contour of* $\mathrm{discr}_{yx}(B)$*;*
- *(ii) the x-coordinate (y-coordinate) of P is a multiple zero of* $\mathrm{discr}_y(B)$ ($\mathrm{discr}_x(B)$).

(b) *Conversely, suppose* $B(x, y; K_1, \ldots, K_q)$ *is such that* $K_1, \ldots, K_q$ *give* $\mathrm{discr}_{yx}(B) = 0$. *Then* $B = 0$ *possesses at least one point* $P = (x', y') \in B_S$*, where* x' *is a multiple zero of* $\mathrm{discr}_y(B)$ *and* y' *is a multiple zero of* $\mathrm{discr}_x(B)$.

(c) *If* $B = 0$ *possesses more than one singular point, these points share no more than two* x*-coordinates and no more than two* y*-coordinates. The maximum number of singular points belonging to any singular curve of* $B = 0$ *is 4.*

Proof.

(a) If $P \in B_S$, then $\mathrm{discr}_{yx}(B)(P) = \left(F_1 B + F_2 \frac{\partial B}{\partial x} + F_3 \frac{\partial B}{\partial y}\right)\Big|_{(x,y)=P} = 0$ follows directly from definition 1 and lemma 3. Statement (i) reiterates that the double discriminant is just a function of the parameters in the biquadratic $B = 0$. For statement (ii), the vanishing of $\mathrm{discr}_{yx}(B)$ ensures that both $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ have a multiple root. But lemma 2 and (20) show that the x-coordinate of P is such a root for $\mathrm{discr}_y(B)$. Similarly, lemma 2 and the corresponding version of (20) for $\frac{\partial}{\partial y}(\mathrm{discr}_x(B))$ show that the y-coordinate of P makes $\mathrm{discr}_x(B)$ and its derivative both vanish.

(b) The given assumption ensures that there are roots x' of $\mathrm{discr}_y(B)$ and y' of $\mathrm{discr}_x(B)$ for which $\frac{\partial}{\partial x}\,\mathrm{discr}_y(B) = 0$ and $\frac{\partial}{\partial y}\,\mathrm{discr}_x(B) = 0$ respectively (i.e. whose multiplicities are greater than 1). In order to establish that $P = (x', y') \in B_S$, we must consider two cases. First, suppose $\alpha_2(x') = 0$. Then (15) gives $\frac{\partial B}{\partial y}(x', y) = 0$ (for any y) and so it follows by (20) that $\frac{\partial \alpha_2}{\partial x}(x')B(x', y) = 0$ and hence also $B(x', y) = 0$ (as $\frac{\partial \alpha_2}{\partial x}(x') = 0$ combined with $\alpha_2(x') = 0$ would contradict assumption 1). Now (15) implies $\frac{\partial B}{\partial x}(x', y') = 0$ and we are done.

Second, suppose $\alpha_2(x') \neq 0$. For $\hat{y} = -\frac{\alpha_1}{2\alpha_2}\Big|_{x=x'}$, we have

$$B(x', \hat{y}) = \alpha_2\left(y + \frac{\alpha_1}{2\alpha_2}\right)^2 - \frac{\mathrm{discr}_y(B)}{4\alpha_2}\Bigg|_{(x,y)=(x',\hat{y})} = 0. \tag{24}$$

Thus (15) provides $\frac{\partial B}{\partial y}(x', \hat{y}) = 0$ and so $\frac{\partial B}{\partial x}(x', \hat{y}) = 0$ (by (20)). Now by (a)(ii) $(x', \hat{y}) \in B_S$ ensures $\hat{y}$ is a multiple root of $\mathrm{discr}_x(B)$, and so we can assume without loss of generality that $\hat{y} = y'$.

(c) Since $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ are quartics in x and y respectively, (a)(ii) ensures that for any combination of $K_1, \ldots, K_q$ associated with a singular curve of $B = 0$, the set of singular points belonging to $B = 0$ must share no more than two x-coordinates and no more than two y-coordinates. Clearly, the maximum cardinality of such a set is 4. □

Remark 2. As noted in the introduction, for fixed parameters the biquadratic (1) is generically an elliptic curve [3, 6], and so can be birationally transformed to the Weierstrass form $Y^2 = X^3 + W_1 X + W_0$. It is shown in [8] that $\mathrm{discr}_{yx}(B) = \Delta/16$, where $\Delta = -16\left(4W_1^3 + 27W_0^2\right)$ is the discriminant of the associated Weierstrass.

Remark 3. It is clear from the proof of proposition 1(b) that *any* point $P = (x', y')$ will be a singular point of $B = 0$ provided either x' and y' are the sole multiple roots of $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ respectively or x' is one of two distinct double roots of $\mathrm{discr}_y(B)$ and y' is the sole multiple root of $\mathrm{discr}_x(B)$; or y' is one of two distinct double roots of $\mathrm{discr}_x(B)$ and x' is the sole multiple root of $\mathrm{discr}_y(B)$. We shall see in section 2 that the only situation in which pairings of multiple roots of these discriminants are not automatically singular is where $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ each possess distinct double roots (see example 4).

The vanishing of $\mathrm{discr}_{yx}(B)$ at any combination of parameters $K_1, \ldots, K_q$ associated with a singular point of (1) restricts the number of singular level sets possessed by $B_{\mathrm{QRT}} = 0$ and $B_{\mathrm{M}} = 0$ to a maximum of 12 and 5, respectively.

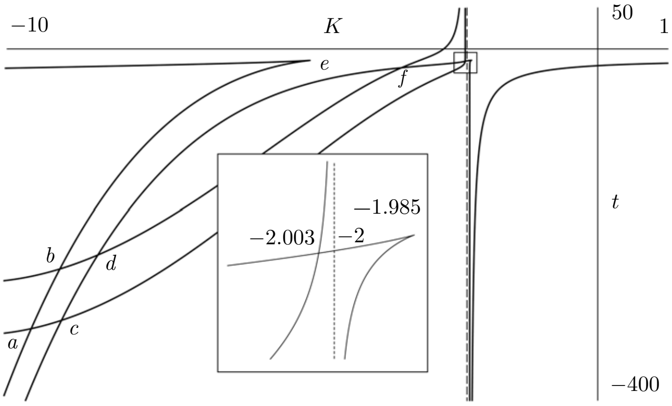


Figure 3. A section of the 0-contour of $\text{discr}_{yx}(B_{\text{M}})$ in $\mathbb{R}^2$ associated with (5). The inset highlights three points (K, t) at which the multiplicity of the t-zeros changes ($K \approx -2.033, -1.985$) or where $\text{discr}_x(\alpha_2) = \text{discr}_y(\beta_2) = -24(K+2)$ vanish. The singular points of the 0-contour labelled a, b, c, d, e and f are associated with singular curves of (5) represented in cases 2, 3, 3, 2, 5 and 2 of table 2 respectively. For example, the parameter combination denoted by e corresponds to the cuspidal singular curve at the bottom of the lower left-hand side of figure 1, whereas f corresponds to the singular curve in the centre plot of figure 2.

Lemma 4. *For each fixed* **K***, (3) possesses no more than 12 singular level sets.*

Proof. It is sufficient to show that when considered as a polynomial in t, $\text{discr}_{yx}(B_{\text{QRT}})$ has maximum degree 12. Let $c_0, \ldots, c_4$ represent the coefficients of $\text{discr}_y(B_{\text{QRT}})$ in x. Since t is affine in each of α_i's of $\text{discr}_y(B_{\text{QRT}}) = \alpha_1^2 - 4\alpha_2\alpha_0$, its presence in each of the coefficients of $\text{discr}_{yx}(B_{\text{QRT}})$ is quadratic. The result follows by counting degrees on the right-hand side of (23). □

The bound for the singular level sets of B_{M} is determined by a similar approach.

Lemma 5. *For each fixed* **K***, (4) possesses no more than five singular level sets.*

Proof. Since the presence of t in B_{M} is affine and isolated to the coefficient α_0, its presence in $\text{discr}_y(B_{\text{M}})$ is also affine and isolated to c_2, c_1 and c_0 (where c_i is defined as the coefficient of x^i in $\text{discr}_y(B_{\text{M}})$). The result follows by counting degrees in (23). □

Example 2. Figure 3 illustrates some of the complexity underlying the McMillan curve family (5). It can be seen that at each $K \in [-10, 1]$, $\text{discr}_{yx}(B_{\text{M}})$ yields five or less zeros in t, only losing or gaining solutions at points (K, t) where the multiplicity of the zeros changes or where the leading coefficient of $\text{discr}_{yx}(B_{\text{M}})$ in t vanishes.

As we have seen, proposition 1 focuses attention on the roots of $\text{discr}_y(B)$ and $\text{discr}_x(B)$ when the biquadratic (1) is singular. The next proposition classifies the various combinations of multiple roots of $\text{discr}_y(B)$ and $\text{discr}_x(B)$ in the singular case, providing necessary conditions for them to occur. To understand the result, we introduce a second global assumption (in addition to assumption 1), a parameter exchange procedure and a range of functions in the coefficients of (1).

With the exception of proposition 5 and cases (b) and (c) of propositions 3 and 4 and the corollaries that follow them, we adopt

Assumption 2.

$$\Phi = \beta(\alpha\epsilon - \beta\delta) - 2\alpha(\alpha\xi - \delta\gamma) \qquad \text{and} \qquad \Psi = \delta(\alpha\epsilon - \beta\delta) - 2\alpha(\alpha\lambda - \beta\kappa) \tag{25}$$

are non-zero.

Definition 3. *The procedure* $VE = CE \circ PE$ *exchanges the elements of* $(x, y, \beta, \delta, \gamma, \kappa, \xi, \lambda)$ *wherever they appear in a given expression with their corresponding elements in* $(y, x, \delta, \beta, \kappa, \gamma, \lambda, \xi)$. *It acts by first applying* PE *to exchange the parameters in the expression, and then applying* CE *to exchange the coordinates.*

Remark 4. Note that $B = 0$ of (1) is invariant under VE. Further, note that when $B = 0$ is a singular curve possessing (up to four) singular points (x_i, y_i), with $1 \leqslant i \leqslant n$ and $n = 1, 2, 3$ or 4 (via proposition 1(c)), the fact that $PE(B) = 0$ is the reflection of $B = 0$ in the line $y = x$ ensures that $PE(B) = 0$ is also a singular curve possessing n singular points, (y_i, x_i). We shall see later, when we come to classify the singular curves of (1), that invoking this symmetry principle means that certain cases follow immediately from others.

The following functions in $\alpha, \ldots, \lambda$ play a central role in the classification of the singular curves of (1):

$$f_{2X} = \mathrm{res}_y(\beta_2, \beta_1) = (\alpha\xi - \delta\gamma)^2 - (\alpha\epsilon - \beta\delta)(\beta\xi - \gamma\epsilon), \tag{26}$$

$$f_{2Y} = \mathrm{res}_x(\alpha_2, \alpha_1) = (\alpha\lambda - \beta\kappa)^2 - (\alpha\epsilon - \beta\delta)(\delta\lambda - \kappa\epsilon), \tag{27}$$

$$f_{2XY} = \mathrm{discr}_x(\alpha_2) f_{2X} - \mathrm{discr}_y(\beta_2) f_{2Y}, \tag{28}$$

$$f_{3XY} \quad \textit{is a polynomial in } \alpha, \ldots, \lambda \textit{ with the leading term } -6912\lambda^4\xi^4\alpha^8 \textit{ in } \alpha. \tag{29}$$

Note that f_{3XY} has not been given explicitly as it consists of 1128 terms (though a formula for it is provided in (78)). Also note that

$$f_{2X} = U_{2X}\beta_2 + V_{2X}\beta_1, \qquad f_{2Y} = U_{2Y}\alpha_2 + V_{2Y}\alpha_1, \tag{30}$$

where

$$\begin{aligned} U_{2X} &= \delta(\alpha\epsilon - \beta\delta)y + \epsilon(\alpha\epsilon - \beta\delta) - \delta(\alpha\xi - \delta\gamma), \\ V_{2X} &= -\alpha(\alpha\epsilon - \beta\delta)y - \beta(\alpha\epsilon - \beta\delta) + \alpha(\alpha\xi - \delta\gamma) \\ \text{and} &\quad U_{2Y} = PE(U_{2X}), V_{2Y} = PE(V_{2X}). \end{aligned}$$

The geometrical significance of the fact that $PE(f_{2X}) = f_{2Y}$ and $PE(f_{3XY}) = f_{3XY}$ will be seen in section 2.

The quantities in (25) are related to f_{2X} and f_{2Y} by

$$\begin{aligned} \Phi^2 &= 4\alpha^2 f_{2X} + \mathrm{discr}_y(\beta_2)(\alpha\epsilon - \beta\delta)^2, \\ \Psi^2 &= 4\alpha^2 f_{2Y} + \mathrm{discr}_x(\alpha_2)(\alpha\epsilon - \beta\delta)^2. \end{aligned} \tag{31}$$

Finally, we lay out a variety of special values that appear frequently in the ensuing proofs and discussions:

$$x_m = \frac{2\alpha\xi + 2\delta\gamma - \beta\epsilon}{2\,\mathrm{discr}_y(\beta_2)}, \qquad y_m = \frac{2\alpha\lambda + 2\beta\kappa - \delta\epsilon}{2\,\mathrm{discr}_x(\alpha_2)} = PE(x_m), \tag{32}$$

$$x^* = \frac{\beta\kappa - \alpha\lambda}{\alpha\epsilon - \beta\delta}, \qquad y^* = \frac{\delta\gamma - \alpha\xi}{\alpha\epsilon - \beta\delta} = PE(x^*), \tag{33}$$

Table 1. The singular curves of (1) can be classified with respect to various combinations of factorizations of $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ as given in (35). (*#SP* stands for the number of singular points.)

Case no	$(a, b, c)\ (d, e, f)$	Necessary conditions on $B = 0$ of (1)	*#SP*
1	(2, 1, 1) (2, 1, 1)	$\mathrm{discr}_{yx}(B) = 0$	1
2	(2, 2, 0) (2, 1, 1)	$\mu = \mu_{d,x}$ and $f_{2X} = 0 or f_{2XY} = 0$	2
3	(2, 1, 1) (2, 2, 0)	$\mu = \mu_{d,y}$ and $f_{2Y} = 0 or f_{2XY} = 0$	2
4	(2, 2, 0) (2, 2, 0)	$\mu = \mu_{d,x} = \mu_{d,y}$ and $f_{2XY} = 0$	2, 3 or 4
5	(3, 1, 0) (3, 1, 0)	$\mu = \mu_t$ and $f_{3XY} = 0$	1
6	(4, 0, 0) (3, 1, 0)	$\mu = \mu_{d,x}$, $f_{2X} = \mathrm{discr}_x(P_{2X}) = 0$ and $f_{2XY} \neq 0$	1
7	(3, 1, 0) (4, 0, 0)	$\mu = \mu_{d,y}$, $f_{2Y} = \mathrm{discr}_y(P_{2Y}) = 0$ and $f_{2XY} \neq 0$	1
8	(4, 0, 0) (4, 0, 0)	$\mu = \mu_{d,x}$ and $f_{2XY} = \mathrm{discr}_x(P_{2X}) = 0$	1

and

$$\mu_{d,x} = \frac{\begin{array}{l}\mathrm{discr}_y(\beta_2)(\beta\epsilon - 2\alpha\xi - 2\delta\gamma)(\epsilon^2 + 2\beta\lambda - 4\delta\xi - 4\gamma\kappa)\\ -2\,\mathrm{discr}_y(\beta_2)^2(\epsilon\lambda - 2\kappa\xi) - (\beta\epsilon - 2\alpha\xi - 2\delta\gamma)^3\end{array}}{4\,\mathrm{discr}_y(\beta_2)\Phi}, \qquad \mu_{d,y} = PE(\mu_{d,x})$$
$$\mu_t = \frac{b_2a_0a_2 - b_0a_2^2 - b_3a_1a_0}{b_1a_2^2 + b_3a_1^2 - b_2a_1a_2 - b_3a_0a_2}, \tag{34}$$

where the coefficients a_i and b_i are given in (70) and (73) respectively.

Proposition 2. *Suppose $B(x_1, y_1; K_1, \ldots, K_q) = 0$ of (1) is singular, i.e. such that $K_1, \ldots, K_q$ satisfy $\mathrm{discr}_{yx}(B) = 0$ by proposition 1(a)(i). Then for these parameter values, the exponents a, b, c, d, e and f in*

$$\begin{aligned}\mathrm{discr}_y(B) &= \mathrm{discr}_y(\beta_2)(x - x_1)^a(x - x_2)^b(x - x_3)^c\\ \mathrm{discr}_x(B) &= \mathrm{discr}_x(\alpha_2)(y - y_1)^d(y - y_2)^e(y - y_3)^f\end{aligned} \tag{35}$$

are such that $2 \leqslant a, d \leqslant 4$ and $0 \leqslant b, c, e, f \leqslant 2$. Necessary conditions for the various factorizations of these discriminants are given in table 1.

Proof. Recall that explicit representations of the quartics $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ are given in (8) and (9), respectively. Throughout this proof we make use of Vieta's well-known formulae relating the coefficients of a quartic $z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$ to its roots (see [1]). In setting

$$z^4 + C_3z^3 + C_2z^2 + C_1z + C_0 = (z - z_1)^2(z - z_2)(z - z_3), \tag{36}$$

we deviate slightly from Vieta's formulation by requiring that the quartic possess at least one multiple root (care of proposition 1(a)(ii)). In this case, we have

$$\begin{aligned}&-2z_1 - z_2 - z_3 = C_3\\ &z_1^2 + 2z_1(z_2 + z_3) + z_2z_3 = C_2\\ &-z_1^2(z_2 + z_3) - 2z_1z_2z_3 = C_1\\ &z_1^2z_2z_3 = C_0.\end{aligned} \tag{37}$$

When applied to $\mathrm{discr}_y(B)/\mathrm{discr}_y(\beta_2)$, the coefficients C_i look like

$$\begin{aligned} C_{x,3} &= \frac{2(\beta\epsilon - 2\alpha\xi - 2\delta\gamma)}{\mathrm{discr}_y(\beta_2)} \\ C_{x,2} &= \frac{\epsilon^2 + 2\beta\lambda - 4\delta\xi - 4\kappa\gamma - 4\alpha\mu}{\mathrm{discr}_y(\beta_2)} \\ C_{x,1} &= \frac{2(\epsilon\lambda - 2\kappa\xi - 2\delta\mu)}{\mathrm{discr}_y(\beta_2)} \\ C_{x,0} &= \frac{\lambda^2 - 4\kappa\mu}{\mathrm{discr}_y(\beta_2)}, \end{aligned} \tag{38}$$

noting that we index with an x to separate these coefficients from their counterparts, $C_{y,i} = PE(C_{x,i})$, belonging to $\mathrm{discr}_x(B)/\mathrm{discr}_x(\alpha_2)$.

Case 1. Observe that the assumption $z^4 + C_3z^3 + C_2z^2 + C_1z + C_0$ possesses a multiple root implies that the quartic's discriminant with respect to z is zero (by (11)). An explicit expression for this discriminant is given on the right-hand side of (23), with $c_i = C_i$ $(i = 0, \dots, 3)$ and $c_4 = 1$. The fact that this discriminant is equal to $\mathrm{discr}_{yx}(B)/\mathrm{discr}_y(B)^6$ when each C_i is replaced with its correspondent in (38) explains why $\mathrm{discr}_{yx}(B) = 0$ in each of the cases of table 1.

Case 2. When $z_1 = x_1$ and $z_2 = z_3 = x_2$ (with $x_1 \neq x_2$) and $C_i = C_{x,i}$ in (37), we have

$$-2(x_1 + x_2) = C_{x,3} \tag{39}$$

$$2x_1x_2 + (x_1 + x_2)^2 = C_{x,2} \tag{40}$$

$$-2x_1x_2(x_1 + x_2) = C_{x,1} \tag{41}$$

$$x_1^2x_2^2 = C_{x,0}, , \tag{42}$$

Combining (39) with (40) on the one hand and (39) with (41) on the other gives expressions for x_1, x_2 in $C_{x,i}$ that together ensure

$$C_{x,3}^3 - 4C_{x,3}C_{x,2} + 8C_{x,1} = 0. \tag{43}$$

Assuming $C_{x,3} \neq 0$, the left-hand side of this equation is affine in μ and solves to give $\mu = \mu_{d,x}$. Further manipulation of (39)–(42) provides

$$C_{x,1}^2 - C_{x,3}^2C_{x,0}\big|_{\mu=\mu_{d,x}} = 0 \quad \Leftrightarrow \quad \frac{64x_m^2 f_{2X} f_{2XY}}{[\mathrm{discr}_y(\beta_2)\Phi]^2} = 0, \tag{44}$$

and this implies

$$f_{2X} f_{2XY} = 0. \tag{45}$$

Finally, it can be shown that

$$\frac{x_1 + x_2}{2} = -\frac{C_{x,3}}{4} = x_m \tag{46}$$

(via (39)) and that x_1, x_2 are the zeros of

$$P'_{2X} = 2C_{x,3}x^2 + C_{x,3}^2x + 2C_{x,1}. \tag{47}$$

When denominators and non-zero factors are cleared and μ is replaced with $\mu_{d,x}$, (47) becomes

$$P_{2X} = a_{2X}x^2 + b_{2X}x + c_{2X} = a_{2X}(x - x_m)^2 - \frac{\mathrm{discr}_x(P_{2X})}{4a_{2X}}, \tag{48}$$

where

$$
\begin{aligned}
a_{2X} &= \mathrm{discr}_y(\beta_2)\Phi \\
b_{2X} &= -2x_m \mathrm{discr}_y(\beta_2)\Phi \\
c_{2X} &= 2\delta f_{2X} + \mathrm{discr}_y(\beta_2)(\lambda(\alpha\epsilon - \beta\delta) - 2\kappa(\alpha\xi - \delta\gamma)).
\end{aligned}
$$

In the case where $C_{x,3} = 0$, $x_1 = -x_2$ by (39) and so $C_{x,1} = 0$ by (41). This gives

$$2\alpha\xi + 2\delta\gamma - \beta\epsilon = 0 \tag{49}$$

$$2\delta\mu + 2\kappa\xi - \epsilon\lambda = 0. \tag{50}$$

Assuming $\alpha \neq 0$ and $\delta \neq 0$, these equations solve to give $\xi = \xi' = (\beta\epsilon - 2\delta\gamma)/2\alpha$ and $\mu = \mu' = (\alpha\epsilon\lambda + 2\delta\gamma\kappa - \beta\kappa\epsilon)/2\alpha\delta = \mu_{d,x}|_{\xi=\xi'}$ and manipulation of (40) and (42) ensures

$$C_{x,2}^2 - 4C_{x,0}\big|_{(\xi,\mu)=(\xi',\mu')} = 0 \Leftrightarrow \left.\frac{16 f_{2X} f_{2XY}}{\delta^2\,\mathrm{discr}_y(\beta_2)^4}\right|_{\xi=\xi'} = 0 \Rightarrow f_{2X} f_{2XY} = 0. \tag{51}$$

Clearly, $x_m = -\frac{C_{x,3}}{4} = 0$ and x_1, x_2 are the zeros of $x^2 + \frac{C_{x,2}}{2}$.

Assuming $\alpha = 0$, we know by assumption 1 that $\beta \neq 0$, $\delta \neq 0$ and so (49) and (50) solve to give $\epsilon = \epsilon' = 2\delta\gamma/\beta$ and $\mu = \mu'' = (\delta\gamma\lambda - \beta\kappa\xi)/\beta\delta = \mu_{d,x}|_{(\alpha,\epsilon)=(0,\epsilon')}$. This ensures

$$C_{x,2}^2 - 4C_{x,0}\big|_{(\alpha,\epsilon,\mu)=(0,\epsilon',\mu'')} = 0 \Leftrightarrow \left.\frac{16 f_{2X} f_{2XY}}{\delta^2\,\mathrm{discr}_y(\beta_2)^4}\right|_{(\alpha,\epsilon)=(0,\epsilon')} = 0 \Rightarrow f_{2X} f_{2XY} = 0. \tag{52}$$

x_m and x_1, x_2 are as in the previous subcase.

Assuming $\delta = 0$ gives $\alpha \neq 0$ and $\kappa \neq 0$ by assumption 1, and (49) and (50) together imply $\xi = \beta\epsilon/2\alpha = \epsilon\lambda/2\kappa \Rightarrow -\epsilon(\alpha\lambda - \beta\kappa)/2\alpha\kappa = 0$. But $\delta = \epsilon = \xi = 0 \Rightarrow \Phi = 0$ and $\delta = \alpha\lambda - \beta\kappa = 0 \Rightarrow \Psi = 0$, both of which contradict assumption 2.

Case 3. A set of coefficient manipulations similar to those above establishes in this case that

$$\mu = \mu_{d,y} = PE(\mu_{d,x}) \tag{53}$$

$$\frac{y_1 + y_2}{2} = y_m = PE(x_m) \tag{54}$$

$$f_{2Y} f_{2XY} = 0, \tag{55}$$

and $P_{2Y}(y_i) = 0$ where $P_{2Y} = VE(P_{2X})$ in the case where $C_{y,3} \neq 0$ and $\left(y^2 + \frac{C_{y,2}}{2}\right)(y_i)$ otherwise.

Case 4. In this case, we know from cases 2 and 3 that $f_{2X} = 0$ or $f_{2XY} = 0$ and $f_{2Y} = 0$ or $f_{2XY} = 0$. By (28) this implies $f_{2XY} = 0$. We also know that $\mu = \mu_{d,x} = \mu_{d,y}$, a relationship confirmed by the easily verified fact that

$$f_{2XY} | (\mu_{d,x} - \mu_{d,y}). \tag{56}$$

We now consider the cases where (36) possesses a triple root in x or y. But before analysing these, we note that $a \geqslant 3 \Leftrightarrow d \geqslant 3$ (meaning that no combination such as $(a, b, c)(d, e, f) = (3, 1, 0)(2, 1, 1)$ appears in table 1). This is explained by the easily verified relationship

$$P_{3Y} = P_{3X} + 12\left(\frac{\partial\alpha_2}{\partial x}\frac{\partial B}{\partial x} - \frac{\partial\beta_2}{\partial y}\frac{\partial B}{\partial y}\right) \tag{57}$$

between the functions

$$
\begin{aligned}
P_{3X} &= \frac{\partial^2}{\partial x^2}\mathrm{discr}_y(B) \\
&= 12\,\mathrm{discr}_y(\beta_2)x^2 - 24\,\mathrm{discr}_y(\beta_2)x_m x + 2(2\beta\lambda + \epsilon^2 - 4(\delta\xi + \gamma\kappa + \alpha\mu))
\end{aligned} \tag{58}
$$

and

$$P_{3Y} = \frac{\partial^2}{\partial y^2}\,\mathrm{discr}_x(B)$$
$$= 12\,\mathrm{discr}_x(\alpha_2)y^2 - 24\,\mathrm{discr}_x(\alpha_2)y_m y + 2(2\delta\xi + \epsilon^2 + -4(\beta\lambda + \gamma\kappa + \alpha\mu)). \tag{59}$$

Now if $a \geqslant 3$ then it must be true that $P_{3X}(x_1) = 0$. Since (x_1, y_1) is singular we also know that $\frac{\partial B}{\partial x}(x_1, y_1) = \frac{\partial B}{\partial y}(x_1, y_1) = 0$. This ensures, via (57), that $P_{3Y}(y_1) = 0$ and hence that $d \geqslant 3$. The fact that $d \geqslant 3 \Rightarrow a \geqslant 3$ is established by a similar argument.

Case 5. When $z_1 = z_2 = x_1, z_3 = x_2$ (with $x_1 \neq x_2$) and $C_i = C_{x,i}$ in (37), we have

$$-3x_1 - x_2 = C_{x,3} \tag{60}$$
$$3x_1^2 + 3x_1x_2 = C_{x,2} \tag{61}$$
$$-x_1^3 - 3x_1^2x_2 = C_{x,1} \tag{62}$$
$$x_1^3x_2 = C_{x,0}. \tag{63}$$

Substituting x_2 of (60) into (61) and (62) gives

$$6x_1^2 + 3C_{x,3}x_1 + C_{x,2} = 0 \tag{64}$$
$$8x_1^3 + 3C_{x,3}x_1^2 - C_{x,1} = 0 \tag{65}$$

respectively. Observe that $8C_{x,2} - 3C_{x,3}^2$ divides the discriminant of the quadratic in (64) with respect to x_1 and so $8C_{x,2} - 3C_{x,3}^2 \neq 0$ else $x_1 = -\frac{C_{x,3}}{4}$, implying via (60) that $x_2 = x_1$.

Equations (64) and (65) now yield

$$x_t = x_1 = \frac{C_{x,3}C_{x,2} - 6C_{x,1}}{8C_{x,2} - 3C_{x,3}^2}, \tag{66}$$

and a similar calculation using the identities corresponding to (60)–(63) for $\mathrm{discr}_x(B)/\mathrm{discr}_x(\alpha_2)$ yields

$$y_t = y_1 = PE(x_t). \tag{67}$$

(Note that $8C_{y,2} - 3C_{y,3}^2 \neq 0$ by the same argument as for $8C_{x,2} - 3C_{x,3}^2$ above, recalling our assumption $y_1 \neq y_2$ for this case.)

We shall now use two equations, (68) and (71), in the coefficients $C_{x,i}$ (which are true by direct substitution of (60)–(63)) to compute μ_t and show that $f_{3XY} = 0$. The identity

$$F'_{x,\mu} = C_{x,2}^2 - 3C_{x,3}C_{x,1} + 12C_{x,0} = 0 \tag{68}$$

is equivalent to

$$\frac{F_{x,\mu}}{\mathrm{discr}_y(\beta_2)^2} = 0, \tag{69}$$

where

$$\begin{aligned} F_{x,\mu} &= 16\alpha^2\mu^2 + 8(-6\kappa\beta^2 - 6\gamma\delta^2 - \alpha\epsilon^2 - 2\alpha\xi\delta - 2\alpha\beta\lambda + 28\gamma\alpha\kappa + 3\delta\beta\epsilon)\mu \\ &\quad + 16\gamma^2\kappa^2 - 48\lambda^2\alpha\gamma - 16\xi\delta\gamma\kappa - 16\lambda\beta\gamma\kappa - 8\gamma\epsilon^2\kappa + 24\lambda\epsilon\delta\gamma - 48\kappa\xi^2\alpha \\ &\quad + 24\lambda\epsilon\xi\alpha + 24\xi\epsilon\kappa\beta + 16\xi^2\delta^2 - 8\epsilon^2\delta\xi - 16\lambda\beta\xi\delta + 16\lambda^2\beta^2 + \epsilon^4 - 8\lambda\beta\epsilon^2 \\ &= a_2\mu^2 + a_1\mu + a_0. \end{aligned} \tag{70}$$

Similarly, the identity

$$G'_{x,\mu} = 54C_{x,3}^3C_{x,1} + 64C_{x,2}^3 + 216C_{x,1}^2 - 18C_{x,3}^2C_{x,2}^2 - 216C_{x,3}C_{x,2}C_{x,1} = 0 \tag{71}$$

is equivalent to

$$\frac{G_{x,\mu}}{\mathrm{discr}_y(\beta_2)^4} = 0, \tag{72}$$

where

$$\begin{aligned} G_{x,\mu} &= 864(\beta\epsilon - 2\alpha\xi - 2\delta\gamma)^3(\epsilon\lambda - 2\kappa\xi - 2\delta\mu) \\ &\quad - 72(\beta\epsilon - 2\alpha\xi - 2\delta\gamma)^2(2\beta\lambda + \epsilon^2 - 4(\delta\xi + \gamma\kappa + \alpha\mu))^2 \\ &\quad - 864\,\mathrm{discr}_y(\beta_2)(\beta\epsilon - 2\alpha\xi - 2\delta\gamma)(2\beta\lambda + \epsilon^2 - 4(\delta\xi + \gamma\kappa + \alpha\mu))(\epsilon\lambda - 2\kappa\xi - 2\delta\mu) \\ &\quad + 64\,\mathrm{discr}_y(\beta_2)(2\beta\lambda + \epsilon^2 - 4(\delta\xi + \gamma\kappa + \alpha\mu))^3 \\ &\quad + 864\,\mathrm{discr}_y(\beta_2)^2(\epsilon\lambda - 2\kappa\xi - 2\delta\mu)^2 \\ &= b_3\mu^3 + b_2\mu^2 + b_1\mu + b_0. \end{aligned} \tag{73}$$

Assuming $\alpha \neq 0$, (69) and (72) solve to give

$$\mu_t = \frac{b_2a_0a_2 - b_0a_2^2 - b_3a_1a_0}{b_1a_2^2 + b_3a_1^2 - b_2a_1a_2 - b_3a_0a_2}. \tag{74}$$

Also, the fact that

$$\mathrm{res}_\mu(F_{x,\mu}, G_{x,\mu}) = -707\,7888\alpha^2\,\mathrm{discr}_y(\beta_2)^2 f_{3XY} \tag{75}$$

ensures, via lemma 1, that there exist expressions U and V for which[6]

$$UF_{x,\mu} + VG_{x,\mu} = -707\,7888\alpha^2\,\mathrm{discr}_y(\beta_2)^2 f_{3XY}, \tag{76}$$

whose vanishing left-hand side (care of (69) and (72)) implies

$$f_{3XY} = 0. \tag{77}$$

Now (75) provides

$$f_{3XY} = -\frac{\begin{array}{c} a_0a_2^2b_1^2 - b_1a_0a_1a_2b_2 - b_1b_0a_1a_2^2 - 2b_1a_0^2b_3a_2 + b_1a_0b_3a_1^2 + a_0^2a_2b_2^2 - 2b_2a_0b_0a_2^2 \\ + b_2b_0a_1^2a_2 - b_2a_0^2b_3a_1 + b_0^2a_2^3 + 3b_0b_3a_0a_1a_2 + a_0^3b_3^2 - b_0b_3a_1^3 \end{array}}{707\,7888\alpha^2\,\mathrm{discr}_y(\beta_2)^2} \tag{78}$$

(noting that $707\,7888\alpha^2\,\mathrm{discr}_y(\beta_2)^2$ divides the numerator of this expression).

In the case where $\alpha = 0$, $\delta \neq 0$ (else $\mathrm{discr}_x(\alpha_2) = 0$) and the vanishing left-hand side of $\mathrm{res}_\mu(F_{x,\mu}, G_{x,\mu})|_{\alpha=0} = \frac{16}{\delta^2} f_{3XY}\big|_{\alpha=0}$ ensures $f_{3XY} = 0$. Further, $F_{x,\mu}|_{\alpha=0}$, being affine in μ, solves to give $\mu = \mu_t|_{\alpha=0}$.

Case 6. When $z_1 = z_2 = z_3 = x_1$ and $C_i = C_{x,i}$ in (37), we have

$$-4x_1 = C_{x,3} \tag{79}$$

$$6x_1^2 = C_{x,2} \tag{80}$$

$$-4x_1^3 = C_{x,1} \tag{81}$$

$$x_1^4 = C_{x,0}. \tag{82}$$

[6] It is easily shown that

$$\begin{aligned} U &= (a_1^2b_3^2 - a_2a_0b_3^2 - a_2a_1b_3b_2 + a_2^2b_3b_1)\mu^2 + (a_1^2b_3b_2 - a_1a_0b_3^2 - a_2a_1b_2^2 + a_2^2b_2b_1 - a_2^2b_3b_0)\mu \\ &\quad + a_0^2b_3^2 - a_1a_0b_3b_2 + a_2a_0b_2^2 + a_1^2b_3b_1 - 2a_2a_0b_3b_1 - a_2a_1b_2b_1 + a_2^2b_1^2 + a_2a_1b_3b_0 - a_2^2b_2b_0 \\ V &= (a_2^2a_0b_3 - a_2a_1^2b_3 + a_2^2a_1b_2 - a_2^3b_1)\mu - a_1^3b_3 + 2a_2a_1a_0b_3 + a_2a_1^2b_2 - a_2^2a_0b_2 - a_2^2a_1b_1 + a_2^3b_0. \end{aligned}$$

Clearly (79) $\Rightarrow x_1 = -\frac{C_{x,3}}{4} = x_m$ and case 2 ensures that x_1 is a double root of P_{2X} (implying $\mathrm{discr}_x(P_{2X}) = 0$), $\mu = \mu_{d,x}$ and $f_{2X} f_{2XY} = 0$. Direct substitution of (79) and (80) ensures $8C_{x,2} - 3C_{x,3}^2 = 0$, which solves to give

$$\mu = \mu' = -\frac{\mathrm{discr}_y(\beta_2)(\epsilon^2 - 4\beta\lambda - 4\delta\xi + 8\gamma\kappa) + 12 f_{2X}}{8\alpha\, \mathrm{discr}_y(\beta_2)}, \tag{83}$$

assuming $\alpha \neq 0$. By case 5 we know $8C_{y,2} - 3C_{y,3}^2 \neq 0$, implying via the identity

$$8C_{y,2} - 3C_{y,3}^2\big|_{\mu=\mu'} = \frac{48 f_{2XY}}{\mathrm{discr}_x(\alpha_2)^2\, \mathrm{discr}_y(\beta_2)}$$

that $f_{2XY} \neq 0$ and hence $f_{2X} = 0$. This reduces (83) to

$$\mu = \mu' = \frac{(4\beta\lambda + 4\delta\xi - \epsilon^2 - 8\gamma\kappa)}{8\alpha}. \tag{84}$$

If $\alpha = 0$, assuming $f_{2XY} = 0$ (which is affine in λ) gives $\lambda = \lambda'$ and thus the contradiction $8C_{y,2} - 3C_{y,3}^2 = 0$ via the identity $\beta^4\delta^2\left(8C_{x,2} - 3C_{x,3}^2\right)\big|_{\lambda=\lambda'} = \beta^2\delta^4\left(8C_{y,2} - 3C_{y,3}^2\right)\big|_{\lambda=\lambda'}$ (recalling $8C_{x,2} - 3C_{x,3}^2 = 0$ and noting that $\beta^2\delta^4 \neq 0$ else either $\alpha = \beta = 0$ or $\alpha = \delta = 0$, contradicting assumption 1).

Case 7. By following arguments identical to those employed in the previous case, it can be shown that when $z_1 = z_2 = z_3 = y_1$ and $C_i = C_{y,i}$ in (37), $y_1 = y_m$, $\mu = \mu_{d,y}$, $\mathrm{discr}_y(P_{2Y}) = 0$ and $f_{2Y} = 0$ but $f_{2XY} \neq 0$.

Case 8. The substitutions $z_1 = z_2 = z_3 = x_1$ and $C_i = C_{x,i}$ in (37) on the one hand and $z_1 = z_2 = z_3 = y_1$ and $C_i = C_{y,i}$ on the other ensure $x_1 = x_m$ and $y_1 = y_m$ in this case. From cases 6 and 7 it is clear that $\mathrm{discr}_x(P_{2X}) = \mathrm{discr}_y(P_{2Y}) = 0$, $\mu = \mu_{d,x} = \mu_{d,y}$, and since $f_{2X} f_{2XY} = 0$ and $f_{2Y} f_{2XY} = 0$ (via cases 2 and 3), we also have $f_{2XY} = 0$. □

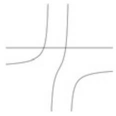
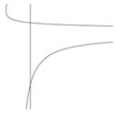
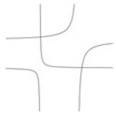
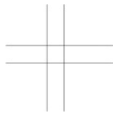
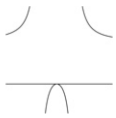
2. Classification of singular curves and their geometry

In this section, we extend and strengthen the classifications in table 1 by providing sufficient conditions for each of the ten possible 'singularity scenarios' associated with $B = 0$. We invoke two corollaries (1 and 2) and several propositions (5)–(11) to prove the classification given in table 2.

In providing necessary and sufficient conditions for each of cases 2–8 in this table, our approach is to specify two types of parameter constraint: one that fixes μ in terms of $\alpha, \ldots, \lambda$ and a set of others in $\alpha, \ldots, \lambda$ (typically involving such expressions as $f_{2X}, f_{2Y}, f_{2XY}, f_{3XY}, \mathrm{discr}_x(P_{2X})$ and $\mathrm{discr}_y(P_{2Y})$) that enables us to target the particular combination of factorizations of $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ concerned. Due to their frequent use in what follows, we remind the reader of where to locate various key quantities. The expressions f_{2X}, f_{2Y}, f_{2XY} are defined in (26)–(28), f_{3XY} in (75), $\mu_{d,x}, \mu_{d,y}$ and μ_t in (34) and (74), and x_m, y_m, x^*, y^* and P_{2X} in (32), (33) and (48) respectively (recalling that $P_{2Y} = VE(P_{2X})$). We also ask the reader to recall assumptions 1 and 2.

Remark 5. Notice that while each of the example curves of $B = 0$—given in the rightmost column of table 2—possesses singular points in $\mathbb{R}^2$, the classification is more general, including curves with affine singular points in $\mathbb{C}^2$ (the $\#SP$ refers to complex affine singular points most generally). Considering that any singular point with some coordinate in $\mathbb{C} - \mathbb{R}$ cannot be 'seen' in the same manner as those depicted in table 2, we have chosen to illustrate the geometric differences between the curves under investigation using examples possessing real singular points only.

Table 2. Conditions on the coefficients $\alpha, \ldots, \mu$ of (1) in order for $B = 0$ to be a singular curve. There are ten classes based on the discriminant factorizations introduced in proposition 2. (#SP stands for the number of singular points.)

Case no.	$(a, b, c)(d, e, f)$	# SP	Necessary and sufficient conditions on $B = 0$ of (1)	Ref.	Example
1	(2, 1, 1) (2, 1, 1)	1	$\mathrm{discr}_{yx}(B) = 0$	Props 1 and 2	
2	(2, 2, 0) (2, 1, 1)	2	$\mu = \mu_{d,x}, f_{2X} = 0, \mathrm{discr}_x(P_{2X}) \neq 0, \alpha\epsilon - \beta\delta \neq 0$	Coro. 1(a), (d), figure 2	
3	(2, 1, 1) (2, 2, 0)	2	$\mu = \mu_{d,y}, f_{2Y} = 0, \mathrm{discr}_y(P_{2Y}) \neq 0, \alpha\epsilon - \beta\delta \neq 0$	Coro. 2(a), (d), figure 4	
4a	(2, 2, 0) (2, 2, 0)	2	$\mu = \mu_{d,x}, f_{2XY} = 0, f_{2X} \neq 0, \mathrm{discr}_x(P_{2X}) \neq 0$	Prop. 7, figure 6	
4b	(2, 2, 0) (2, 2, 0)	3	$\mu = \mu_{d,x}, f_{2X} = f_{2Y} = 0, \mathrm{discr}_x(P_{2X}) \neq 0, \alpha\epsilon - \beta\delta \neq 0$	Prop. 6, figure 5	
4c	(2, 2, 0) (2, 2, 0)	4	$\mu = \frac{\gamma\kappa}{\alpha}, f_{2X} = f_{2Y} = \alpha\epsilon - \beta\delta = 0, \alpha \neq 0$	Prop. 5	
5	(3, 1, 0) (3, 1, 0)	1	$\mu = \mu_t, f_{3XY} = 0, 8C_{x,2} - 3C_{x,3}^2 \neq 0, 8C_{y,2} - 3C_{y,3}^2 \neq 0$	Prop. 8, figure 7	
6	(4, 0, 0) (3, 1, 0)	1	$\mu = \mu_{d,x}, f_{2X} = \mathrm{discr}_x(P_{2X}) = 0, f_{2XY} \neq 0$	Prop. 9	
7	(3, 1, 0) (4, 0, 0)	1	$\mu = \mu_{d,y}, f_{2Y} = \mathrm{discr}_y(P_{2Y}) = 0, f_{2XY} \neq 0$	Prop. 10	
8	(4, 0, 0) (4, 0, 0)	1	$\mu = \mu_{d,x}, f_{2XY} = \mathrm{discr}_x(P_{2X}) = 0$	Prop. 11	

Remark 6. The biquadratics under review have degree $d = 4$ and, as noted in remark 1, possess two singular points at infinity (when considered projectively). Thus if a given curve contains one affine singular point, it will have genus $g = \frac{1}{2}(d-1)(d-2) - \#SP = 0$ (as in

case 1 of table 2). When $\#SP \geqslant 2$, however, the fact that the genus of an irreducible algebraic curve must be non-negative implies that the biquadratic will be reducible, i.e. it will factor into a product curve. This is evident in cases 2–4 of table 2.

Remark 7. We remind the reader of definition 3 of section 1 and remark 4 showing that $PE(B) = 0$ is the reflection of $B = 0$ in the line $y = x$. This symmetry principle obviates the need to prove every result represented in table 2 (specifically, it provides that propositions 4 and 10 and corollary 2 follow from propositions 3 and 9 and corollary 1 respectively).

Remark 8. Before beginning the analysis to establish table 2, we consider the special case when $B = 0$ is McMillan (4) and return to figures 1 and 2 where particular cases of table 2 were illustrated (also see figure 3). The expressions f_{2X}, f_{2Y}, f_{2XY} and f_{3XY} of (26)–(29) are all expressions in $\mathbf{K} = \{K_2, \ldots, K_q\}$. Considering that the 0-contour of $\mathrm{discr}_{yx}(B_{\mathrm{M}})$ is a hypersurface in $\mathbb{C}^q$ comprising the biquadratic's singular parameter combinations (via proposition 1), it is no coincidence that the singularities of $\mathrm{discr}_{yx}(B_{\mathrm{M}}) = 0$, in turn, can be related to f_{2X}, f_{2Y}, f_{2XY} and f_{3XY} (as the vanishing of at least one of these expressions is a condition for cases 2–8 in table 2). In fact, it is easily shown that the discriminant with respect to t of the quintic[7], $\mathrm{discr}_{yx}(B_{\mathrm{M}})$, in t is

$$\mathrm{discr}_{yxt}(B_{\mathrm{M}}) = 184\,467\,440\,737\,095\,516\,16 f_{2X}^2 f_{2Y}^2 f_{2XY}^2 f_{3XY}^3. \tag{85}$$

This means that if $\mathbf{K} = \mathbf{K}^*$ is such that one of f_{2X}, f_{2Y}, f_{2XY} or f_{3XY} is zero, it is always possible to find a pair $(\mathbf{K}, t) = (\mathbf{K}^*, t^*)$ for which $\mathrm{discr}_{yx}(B_{\mathrm{M}}) = \frac{\partial}{\partial t}\mathrm{discr}_{yx}(B_{\mathrm{M}}) = 0$. It is an interesting fact that in all of the examples we have considered, the vanishing of these polynomials has guaranteed the vanishing of $\frac{\partial}{\partial K_2}\mathrm{discr}_{yx}(B_{\mathrm{M}}), \ldots, \frac{\partial}{\partial K_q}\mathrm{discr}_{yx}(B_{\mathrm{M}})$, meaning that $(\mathbf{K}^*, t^*)$ represents a singular point of the contour $\mathrm{discr}_{yx}(B_{\mathrm{M}}) = 0$ (seen as one of the self-intersection points in figure 3, for example).

We begin by presenting two results that help to explain the geometry of $B = 0$ when it contains more than one point of horizontal (lemma 6) or vertical (lemma 7) tangency. We shall see that in each case the biquadratic decomposes to include a linear component (or subset, specified by the factor $(y - y')$ in the former case) and another, nonlinear, component (which may itself decompose to include further linear components).

The proof of lemma 7 is similar to that of lemma 6 and is omitted.

Lemma 6.

(a) *Suppose the biquadratic $B = 0$ of (1) contains the points $P_1 = (x_1, y')$ and $P_2 = (x_2, y')$ with $x_1 \neq x_2$ and $\frac{\partial B}{\partial x}(P_i) = 0$. Then $f_{2X} = 0$ and*

$$B = (y - y')(\alpha_2(y + y') + \alpha_1). \tag{86}$$

(b) *If there is a $y = y'$ such that $\beta_2(y') = \beta_1(y') = \beta_0(y') = 0$, then any point $P = (x, y')$ ensures $B(P) = \frac{\partial B}{\partial x}(P) = 0$.*

Proof.

(a) First observe that

$$\begin{aligned} \frac{\partial B}{\partial x}(x_1, y') = 0 \quad &\Leftrightarrow \quad 2\beta_2(y')x_1 + \beta_1(y') = 0 \\ \frac{\partial B}{\partial x}(x_2, y') = 0 \quad &\Leftrightarrow \quad 2\beta_2(y')x_2 + \beta_1(y') = 0 \end{aligned} \tag{87}$$

[7] See lemma 5.

together imply $\beta_2(y') = 0$ and hence $\beta_1(y') = \beta_0(y') = 0$ also (using $B(P_i) = 0$ for the latter). Recalling (6) it is clear that any point (x, y') lies on $B = 0$ and hence that $B = (y - y')(\alpha_2(y + y') + \alpha_1)$ (by the factor theorem). Further, we know by (30) that $f_{2X} = 0$.

(b) As for (a) $\beta_2(y') = \beta_1(y') = \beta_0(y') = 0$ implies that B is of the form (86) and clearly this guarantees $B(P) = \frac{\partial B}{\partial x}(P) = 0$ for any $P = (x, y')$. □

Lemma 7.

(a) Suppose the biquadratic $B = 0$ of (1) contains the points $P_1 = (x', y_1)$ and $P_2 = (x', y_2)$ with $y_1 \neq y_2$ and $\frac{\partial B}{\partial y}(P_i) = 0$. Then $f_{2Y} = 0$ and

$$B = (x - x')(\beta_2(x + x') + \beta_1). \tag{88}$$

(b) If there is an $x = x'$ such that $\alpha_2(x') = \alpha_1(x') = \alpha_0(x') = 0$, then any point $P = (x', y)$ ensures $B(P) = \frac{\partial B}{\partial y}(P) = 0$.

Proposition 3. *The biquadratic $B = 0$ of (1) contains distinct points $P_1 = (x_1, y')$ and $P_2 = (x_2, y')$ for which $\frac{\partial B}{\partial x}(P_i) = 0$ if and only if $f_{2X} = 0$ and*

(a) $\mu = \mu_{a,x} = \frac{\gamma(\beta\lambda - \gamma\kappa)}{\beta^2}$, assuming $\alpha = 0$,

(b) μ is strictly one of $\mu^{\pm}_{b,x} = \frac{(\alpha\lambda-\beta\kappa)(\beta\mp\sqrt{\mathrm{discr}_y(\beta_2)})+2\alpha\gamma\kappa}{2\alpha^2}$, assuming $\alpha \neq 0, \alpha\epsilon - \beta\delta = 0$ and $\alpha\lambda - \beta\kappa \neq 0$,

(c) $\mu = \mu_{c,x} = \frac{\gamma\kappa}{\alpha}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\lambda - \beta\kappa = 0$,

(d) $\mu = \mu_{d,x}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$.

Proof. For the necessary direction in each case, we assume P_1 and P_2 are defined as above. Then lemma 6(a) ensures $f_{2X} = 0$ and $\beta_2(y') = \beta_1(y') = \beta_0(y') = 0$. To prove the sufficient direction, we specify a $y = y'$ which combines with the given parameter constraints to ensure $\beta_2(y') = \beta_1(y') = \beta_0(y') = 0$. The result will then follow by lemma 6(b).

(a) Assume $\alpha = 0$ and note that neither β nor δ are zero in this case, else $\mathrm{discr}_y(\beta_2) = \beta^2 - 4\alpha\gamma = 0$ or $\mathrm{discr}_x(\alpha_2) = \delta^2 - 4\alpha\kappa = 0$, contradicting assumption 1. Now $\beta_2(y') = 0 \Rightarrow y' = y_a = -\frac{\gamma}{\beta}$, and solving for μ in $\beta_0(y') = 0$ gives $\mu = \mu_{a,x}$.

Conversely, suppose $\mu = \mu_{a,x}$ and $f_{2X} = 0$. For $y' = y_a$, we know from above that $\beta_0(y') = 0$. Also, $\beta_2(y') = \frac{\alpha\gamma^2}{\beta^2} = 0$ and $\beta_1(y') = \frac{f_{2X}|_{\alpha=0}}{\delta\beta^2} = 0$.

(b) Assume $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = 0$ but $\alpha\lambda - \beta\kappa \neq 0$. When the y-zeros, $y' = y_b^{\pm}$, of β_2 ('+' denoting the '$+\sqrt{\ }$' solution) are substituted into $\beta_0(y') = 0$, the μ-solutions corresponding to them are $\mu^{\pm}_{b,x}$ (where y_b^+ is paired with $\mu^+_{b,x}$)[8]. This means that μ is one of $\mu^+_{b,x}$ or $\mu^-_{b,x}$ but not both due to

$$\mu^+_{b,x} - \mu^-_{b,x} = -\frac{(\alpha\lambda - \beta\kappa)\sqrt{\mathrm{discr}_y(\beta_2)}}{\alpha^2} \neq 0. \tag{89}$$

Conversely, suppose μ is one of $\mu^+_{b,x}$ or $\mu^-_{b,x}$ and $f_{2X} = 0$. From above we know that $\beta_0\big|_{\mu=\mu^{\pm}_{b,x}}\left(y_b^{\pm}\right) = \beta_2\left(y_b^{\pm}\right) = 0$. Also,

$$\beta_1\left(y_b^{\pm}\right) = \frac{(\alpha\epsilon - \beta\delta)\{\pm\sqrt{\mathrm{discr}_y(\beta_2)} - \beta\} + 2\alpha(\alpha\xi - \delta\gamma)}{2\alpha^2} = 0,$$

noting that $f_{2X} = \alpha\epsilon - \beta\delta = 0 \Rightarrow \alpha\xi - \delta\gamma = 0$ by (26).

[8] The identity $B|_{\mu^{\pm}_{b,x}} = (y - y_b^{\mp})(\alpha_2 y + \alpha_2 y_b^{\mp} + \alpha_1) - \frac{\{(\alpha\epsilon-\beta\delta)(\beta\mp\sqrt{\mathrm{discr}_y(\beta_2)})-2\alpha(\alpha\xi-\delta\gamma)\}x\pm 2(\alpha\lambda-\beta\kappa)\sqrt{\mathrm{discr}_y(\beta_2)}}{2\alpha^2} = (y - y_b^{\mp})(\alpha_2 y + \alpha_2 y_b^{\mp} + \alpha_1) \mp \frac{2(\alpha\lambda-\beta\kappa)\sqrt{\mathrm{discr}_y(\beta_2)}}{2\alpha^2}$ makes it clear that the combinations $\mu^+_{b,x}, y_b^-$ and $\mu^-_{b,x}, y_b^+$ are not possible, else $\alpha\lambda - \beta\kappa = 0$.

(c) Assume $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\lambda - \beta\kappa = 0$. Then an argument similar to that in (b) establishes $\mu = \mu_{c,x} = \mu_{b,x}^{+} = \mu_{b,x}^{-} = \frac{\gamma\kappa}{\alpha}$.
The converse is also true by (b). In this case, B's representation is very simple:

$$B|_{\mu=\mu_{c,x}} = \frac{\alpha_2\beta_2}{\alpha}. \tag{90}$$

This is due to $B|_{\mu=\mu_{c,x}}\left(x, y_c^{\pm}\right) = \frac{\partial B}{\partial x}\left(x, y_c^{\pm}\right) = 0$, implying that $\left(y - y_c^{+}\right)$ and $\left(y - y_c^{-}\right)$ are distinct factors of B.

(d) Assume $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$. Then $\beta_2(y') = \beta_1(y') = 0$ solves to give $y = y^*$ and $\beta_0(y^*) = 0$ implies

$$\mu = \mu_2 = \frac{(\alpha\xi - \delta\gamma)(\lambda(\alpha\epsilon - \beta\delta) - \kappa(\alpha\xi - \delta\gamma))}{(\alpha\epsilon - \beta\delta)^2}. \tag{91}$$

That $\mu_2 = \mu_{d,x}$ in this case is due to the easily verified identity

$$\mu_2 - \mu_{d,x} = \frac{f_{2X}\{[\epsilon(\alpha\epsilon-\beta\delta)-\beta(\alpha\lambda-\kappa\beta)-\alpha(\beta\lambda-4\kappa\gamma)]\Phi+[2\alpha(\beta\xi-\gamma\epsilon)-2\gamma(\alpha\epsilon-\beta\delta)]\Psi\}}{\Phi\mathrm{discr}_y(\beta_2)(\alpha\epsilon-\beta\delta)^2}. \tag{92}$$

Conversely, suppose $f_{2X} = 0$ and $\mu = \mu_{d,x} = \mu_2$ (by (92)). Then for $y' = y^*$, we know by the above that $\beta_0|_{\mu=\mu_{d,x}}(y') = \beta_0|_{\mu=\mu_2}(y') = 0$. Also, $\beta_2(y') = \frac{\alpha f_{2X}}{(\alpha\epsilon-\beta\delta)^2} = 0$ and $\beta_1(y') = \frac{\delta f_{2X}}{(\alpha\epsilon-\beta\delta)^2} = 0$. □

Note that when both of the factors in (86) are zero, $(2\alpha_2 y' + \alpha_1)(P_i) = \frac{\partial B}{\partial y}(P_i) = 0$ and so P_1 and P_2 are singular. Also observe that assumption 2 does not apply in cases (a) and (b) as here $\Phi = 0$ but $\Psi \neq 0$ and $\Phi = \Psi = 0$ respectively, via (31).

Corollary 1. *The biquadratic $B = 0$ of (1) contains two distinct singular points $P_1 = (x_1, y')$ and $P_2 = (x_2, y')$ if and only if $f_{2X} = 0$ and*

(a) (Case 2) $\mu = \mu_{a,x} = \frac{\gamma(\beta\lambda-\gamma\kappa)}{\beta^2}$ and $\mathrm{discr}_x(P_{2X}) \neq 0$, assuming $\alpha = 0$,

(b) μ is strictly one of $\mu_{b,x}^{\pm} = \frac{(\alpha\lambda-\beta\kappa)(\beta\pm\sqrt{\mathrm{discr}_y(\beta_2)})+2\alpha\gamma\kappa}{2\alpha^2}$ but $\mathrm{discr}_x\left(\frac{\partial B}{\partial y}\big|_{y=y_b^+}\right) \neq 0$ in the case where $\mu = \mu_{b,x}^{+}$ or $\mathrm{discr}_x\left(\frac{\partial B}{\partial y}\big|_{y=y_b^-}\right) \neq 0$ in the case where $\mu = \mu_{b,x}^{-}$, assuming $\alpha \neq 0, \alpha\epsilon - \beta\delta = 0$ and $\alpha\lambda - \beta\kappa \neq 0$,

(c) $\mu = \mu_{c,x} = \frac{\gamma\kappa}{\alpha}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\lambda - \beta\kappa = 0$,

(d) (Case 2) $\mu = \mu_{d,x}$ and $\mathrm{discr}_x(P_{2X}) \neq 0$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$.

Proof. For the necessary direction, we only need to show (thanks to proposition 3) that $\mathrm{discr}_x(P_{2X}) \neq 0$ in cases (a) and (d) and that $\mathrm{discr}_x\left(\frac{\partial B}{\partial y}\big|_{y=y_b^{\pm}}\right) \neq 0$ when $\mu = \mu_{b,x}^{\pm}$ in case (b). To prove the sufficient direction, it is enough to specify a $y = y'$ and distinct x_1, x_2 for which $\frac{\partial B}{\partial y}(x_i, y') = 0$ (again thanks to proposition 3).

(a) Assume $\alpha = 0$. Then case 2 of proposition 2 ensures that x_1, x_2 are the zeros of P_{2X}, whose distinctness guarantees $\mathrm{discr}_x(P_{2X}) \neq 0$.

Conversely, suppose $\mu = \mu_{a,x}$, $f_{2X} = 0$ and $\mathrm{discr}_x(P_{2X}) \neq 0$. When $y' = y_a$ and x_1, x_2 are the distinct zeros of P_{2X}, we have $\frac{\partial B}{\partial y}(x_i, y') = \frac{2 f_{2X}|_{\alpha=0}}{\beta^3} = 0$.

(b) Assume $\alpha \neq 0, \alpha\epsilon - \beta\delta = 0$ and $\alpha\lambda - \beta\kappa \neq 0$. In the case where $\mu = \mu_{b,x}^{+}$ and $y' = y_b^{+}$, the fact that x_1, x_2 are the distinct zeros of the quadratic, $\frac{\partial B}{\partial y}\big|_{y=y_b^+}$, in x ensures $\mathrm{discr}_x\left(\frac{\partial B}{\partial y}\big|_{y=y_b^+}\right) \neq 0$.

Conversely, if $\mu = \mu_{b,x}^{+}$ and $f_{2X} = 0$ but $\mathrm{discr}_x\left(\frac{\partial B}{\partial y}\big|_{y=y_b^+}\right) \neq 0$, then when $y' = y_b^{+}$ and x_1, x_2 are the distinct zeros of $\frac{\partial B}{\partial y}\big|_{y=y_b^+}$, we clearly have $\frac{\partial B}{\partial y}\left(x_i, y_b^{+}\right) = 0$. A similar argument applies for the $\mu_{b,x}^{-}, y_b^{-}$ pairing.

(c) Assume $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\lambda - \beta\kappa = 0$. The necessary direction is already covered by proposition 3, so for the converse we suppose $\mu = \mu_{c,x}$, $f_{2X} = 0$, $y' = y_b^{\pm}$ and let $x_b^{\pm}$ be the x-zeros of α_2 (distinct due to assumption 1, with '+' denoting the '$+\sqrt{\ }$' solution). By (90), we know that $B|_{\mu=\mu_{c,x}} = \frac{\alpha_2\beta_2}{\alpha}$ and so clearly $\frac{\partial B}{\partial y}\left(x_b^{\pm}, y'\right) = \frac{\alpha_2(x_b^{\pm})}{\alpha}\frac{\partial\beta_2}{\partial y}(y') = 0$.

(d) Assume $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$. The fact that $\mathrm{discr}_x(P_{2X}) \neq 0$ is as in (a).

Conversely, suppose $f_{2X} = 0$, $\mu = \mu_{d,x} = \mu_2$ (by (92)) and $\mathrm{discr}_x(P_{2X}) \neq 0$. Let x_1, x_2 be the distinct zeros of P_{2X} and $y' = y^*$. Then since

$$\left.\frac{\partial B}{\partial y}\right|_{y=y^*} = \frac{P_{2X} - 2\frac{\partial\alpha_2}{\partial x}f_{2X}}{(\alpha\epsilon - \beta\delta)\,\mathrm{discr}_y(\beta_2)} = \frac{P_{2X}}{(\alpha\epsilon - \beta\delta)\,\mathrm{discr}_y(\beta_2)}, \tag{93}$$

it is clear that $\frac{\partial B}{\partial y}(x_i, y') = 0$. □

Proposition 4 and corollary 2 are a straightforward consequence of proposition 3 and corollary 1 and the symmetry principle noted in remark 6.

Proposition 4. *The biquadratic $B = 0$ of (1) contains distinct points $P_1 = (x', y_1)$ and $P_2 = (x', y_2)$ for which $\frac{\partial B}{\partial y}(P_i) = 0$ if and only if $f_{2Y} = 0$ and*

(a) $\mu = \mu_{a,y} = PE(\mu_{a,x})$, assuming $\alpha = 0$,
(b) μ is strictly one of $\mu_{b,y}^{\pm} = PE\left(\mu_{b,x}^{\pm}\right)$, assuming $\alpha \neq 0, \alpha\epsilon - \beta\delta = 0$ and $\alpha\xi - \delta\gamma \neq 0$,
(c) $\mu = \mu_{c,y} = \frac{\gamma\kappa}{\alpha}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\xi - \delta\gamma = 0$,
(d) $\mu = \mu_{d,y}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$.

Corollary 2. *The biquadratic $B = 0$ of (1) contains two distinct singular points $P_1 = (x', y_1)$ and $P_2 = (x', y_2)$ if and only if $f_{2Y} = 0$ and*

(a) (Case 3) $\mu = \mu_{a,y} = PE(\mu_{a,x})$ but $\mathrm{discr}_y(P_{2Y}) \neq 0$, assuming $\alpha = 0$,[8]
(b) μ is strictly one of $\mu_{b,y}^{\pm} = PE\left(\mu_{b,x}^{\pm}\right)$ but $\mathrm{discr}_y\left(\left.\frac{\partial B}{\partial x}\right|_{x=x_b^+}\right) \neq 0$ in the case where $\mu = \mu_{b,y}^{+}$ or $\mathrm{discr}_y\left(\left.\frac{\partial B}{\partial x}\right|_{x=x_b^-}\right) \neq 0$ in the case where $\mu = \mu_{b,y}^{-}$, assuming $\alpha \neq 0, \alpha\epsilon - \beta\delta = 0$ and $\alpha\xi - \delta\gamma \neq 0$,
(c) $\mu = \mu_{c,y} = \frac{\gamma\kappa}{\alpha}$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta = \alpha\xi - \delta\gamma = 0$,
(d) (Case 3) $\mu = \mu_{d,y}$ and $\mathrm{discr}_y(P_{2Y}) \neq 0$, assuming $\alpha \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$.

Example 3. The McMillan biquadratic (5) yields

$$f_{2Y} = \frac{746\,3287}{156\,25} + \frac{348\,72201}{312\,50}K + \frac{579\,447\,33}{625\,00}K^2 + \frac{403\,715\,31}{125\,000}K^3 + \frac{134\,991}{3125}K^4 + \frac{240\,267}{125\,000}K^5, \tag{94}$$

which vanishes at $K \approx -2.0029$. We also have $\mu = \mu_{d,y} \approx 0.5287$, giving $t = \mu_{\mathrm{M}} - \mu \approx -1.5317$ (see (4)), $\mathrm{discr}_y(P_{2Y}) \approx 3.1520$, $\alpha = 2$ and $\alpha\epsilon - \beta\delta \approx 6.0526$. Figure 4 illustrates the fact that the line $x = x^* \approx 0.0661$, which contains the two singularities $P_1 \approx (0.0661, 2.2152)$ and $P_2 \approx (0.0661, -13.6632)$, is a subset of the singular level set[9].

Proposition 5 (*Case 4c*)**.** *The biquadratic $B = 0$ of (1) contains exactly four distinct singular points if and only if $\alpha \neq 0$, $\mu = \frac{\gamma\kappa}{\alpha}$ and $\alpha\epsilon - \beta\delta = f_{2X} = f_{2Y} = 0$.*

Proof. First note that by proposition 1(c), the maximum possible number of singular points possessed by $B = 0$ is 4. Suppose $P_1, P_2, P_3, P_4 \in B_S$ are distinct. By proposition 2

[9] That the four asymptotes (see the discussion preceding lemma 8) associated with $B_{\mathrm{M}} = 0$ in this example are given by $x = \pm x^*$ and $y = \pm x^*$ is due to the atypical nature of the biquadratic: $\beta = \delta = 0$ and $\gamma = \kappa$ ensure that when y is replaced with x in β_2, $\alpha_2 = \beta_2$.

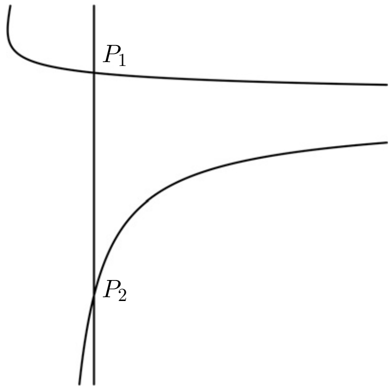


Figure 4. Illustration of case 3 of table 2 (see corollary 2(d)).

these points share exactly two x-coordinates and two y-coordinates, meaning that we can put $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_1)$, $P_3 = (x_2, y_2)$ and $P_4 = (x_1, y_2)$ with $x_1 \neq x_2, y_1 \neq y_2$. By lemmas 6(a) and 7(a) we know that $f_{2X} = f_{2Y} = 0$ and $\beta_i(y_1) = \beta_i(y_2) = 0, i = 0, 1, 2$. Manipulating the latter gives $(y_1 - y_2)[\alpha(y_1 + y_2) + \beta] = 0 \Rightarrow \alpha \neq 0$ (else $\beta = \mathrm{discr}_y(\beta_2) = 0$ contradicting assumption 1). Also, eliminating y_1 and y_2 gives $\alpha\epsilon - \beta\delta$ and hence $\alpha\lambda - \beta\kappa = 0$ by (27). Now corollary 1(c) ensures $\mu = \frac{\gamma\kappa}{\alpha}$.

Conversely, suppose $\alpha \neq 0$, $\mu = \frac{\gamma\kappa}{\alpha}$ and $\alpha\epsilon - \beta\delta = f_{2X} = f_{2Y} = 0$. The latter implies $\alpha\lambda - \beta\kappa = 0$ by (27) and hence $B = \frac{\alpha_2\beta_2}{\alpha}$ by the proof of corollary 1(c). This ensures that when $x_1 = x_b^+, x_2 = x_b^-, y_1 = y_b^+$ and $y_2 = y_b^-$ (the coordinate-wise distinct zeros of α_2 and β_2 respectively), $P_i \in B_S$. □

Proposition 6 (*Case 4b*)**.** *The biquadratic $B = 0$ of (1) contains exactly three distinct singular points if and only if $\mu = \mu_{d,x}$, $f_{2X} = f_{2Y} = 0$, $\mathrm{discr}_x(P_{2X}) \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$.*

Proof. Suppose $P_1, P_2, P_3 \in B_S$ are distinct. By proposition 1(c) these points are made up of exactly two x-coordinates and two y-coordinates, meaning that we can put $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_1)$ and $P_3 = (x_1, y_2)$, with $x_1 \neq x_2$ and $y_1 \neq y_2$. It follows by corollaries 1 and 2 that $f_{2X} = f_{2Y} = 0$. We consider two cases on α.

If $\alpha = 0$ then $\mu = \mu_{d,x}$ by corollary 1(a), noting that $\mu_{a,x} = \mu_2|_{\alpha=0} = \mu_{d,x}|_{\alpha=0}$ by (92). Further, $\alpha\epsilon - \beta\delta \neq 0$ else either $\beta = \mathrm{discr}_y(\beta_2) = 0$ or $\alpha = \mathrm{discr}_x(\alpha_2) = 0$, contradicting assumption 1. If $\alpha \neq 0$ then assuming $\alpha\epsilon - \beta\delta = 0$ gives $\alpha\lambda - \beta\kappa = 0$ by (27) and hence $\mu = \mu_{c,x}$ by corollary 1(c). But this implies the existence of a fourth singular point distinct from P_1, P_2 and P_3 (via proposition 5), which is a contradiction. Thus $\alpha\epsilon - \beta\delta \neq 0$ and corollary 1(d) provides $\mu = \mu_{d,x}$. The fact that $\mathrm{discr}_x(P_{2X}) \neq 0$ in both cases is due to (a) and (d) of corollary 1.

Conversely, suppose $\mu = \mu_{d,x}$, $f_{2X} = f_{2Y} = 0$, $\mathrm{discr}_x(P_{2X}) \neq 0$ and $\alpha\epsilon - \beta\delta \neq 0$. First observe that since $f_{2XY} = 0$ by (28), $\mu_{d,x} = \mu_{d,y}$ by (56). It follows by propositions 3(d) and 4(d) in combination with lemmas 6(b) and 7(b) that B factorizes to become

$$B = (x - x^*)(y - y^*)(\alpha xy + (\alpha y^* + \beta)x + (\alpha x^* + \delta)y + \alpha x^* y^* + \beta x^* + \delta y^* + \epsilon). \tag{95}$$

Thus $\frac{\partial B}{\partial x} = (y - y^*)Q$ with $Q = 2\alpha xy + 2(\alpha y^* + \beta)x + \delta y + \delta y^* + \epsilon$ and $\frac{\partial B}{\partial y} = VE\left(\frac{\partial B}{\partial x}\right)$, and so considering that $Q(x^*, 2y_m - y^*) = -\frac{4f_{2Y}}{\mathrm{discr}_x(\alpha_2)(\alpha\epsilon - \beta\delta)} = 0$ and similarly $VE(Q) = 0$, it is clear that each of the points $P_1 = (x^*, y^*)$, $P_2 = (x^*, 2y_m - y^*)$ and $P_3 = (2x_m - x^*, y^*)$ belongs to B_S. Proposition 2 now ensures that $x^*, 2x_m - x^*$ are the zeros of P_{2X}, distinct due

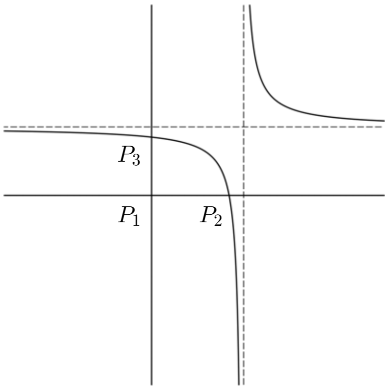


Figure 5. Illustration of case 4b of table 2 (see proposition 6). Since for any $B = 0$ satisfying the conditions of proposition 6, $\alpha_2(x^*) = \frac{\alpha f_{2Y}}{(\alpha\epsilon-\beta\delta)^2} = 0$ and $\beta_2(y^*) = \frac{\alpha f_{2X}}{(\alpha\epsilon-\beta\delta)^2} = 0$, it is clear that two of the four asymptotes of the biquadratic (see the discussion preceding lemma 8) are wholly contained in the singular curve.

to $\mathrm{discr}_x(P_{2X}) \neq 0$ and that $y^*, 2y_m - y^*$ are the zeros of P_{2Y}, distinct due to $\mathrm{discr}_y(P_{2Y}) \neq 0$ using the identity

$$\mathrm{discr}_x(\alpha_2)^2\,\mathrm{discr}_x(P_{2X}) - \mathrm{discr}_y(\beta_2)^2\,\mathrm{discr}_y(P_{2Y}) = P f_{2XY}, \tag{96}$$

where P is a 32-term expression in $\alpha, \ldots, \lambda$ with the leading term $64(\kappa\xi^2 - \gamma\lambda^2)\alpha^5$ in α.

The existence of a fourth singular point not equal to any P_i would contradict our assumption that $\alpha\epsilon - \beta\delta \neq 0$ (via proposition 5). □

Example 4. Taking the resultant with respect to K_2 of f_{2X} and f_{2Y} associated with the biquadratic

$$\begin{aligned} B &= (2K_2+1)x^2y^2 + \left(K_2{}^3K_3 - K_2{}^2 - 2K_2 + 1\right)x^2y + xy^2 + (K_2{}^3 + 9K_2{}^2 + 27K_2 + 12)x^2 \\ &\quad + (3K_2c + 6)y^2 + xy + (K_2+1)x - \left(8K_2{}^3 + 42K_2{}^2 + 36K_2 - 6\right)y + 17K_2{}^3 \\ &\quad + 24K_2{}^2 + 87K_2 + 3 - t \end{aligned} \tag{97}$$

gives a degree-42 polynomial in K_3, one of whose zeros is $K_3 \approx -0.8599$ (noting that both f_{2X} and f_{2XY} are bivariate in K_2 and t).

Back substitution provides $K_2 \approx -2.9401$ and using $\mathrm{discr}_{yx}(B) = 0$ gives $t \approx -511.2514$. We also have $\mu = \mu_{d,x} = \mu_{d,y} \approx 33.8870$. Figure 5 illustrates that $B = 0$ contains the lines $x = x^* \approx -1.5691$ and $y = y^* \approx 0.9799$ and the three singularities $P_1 = (x^*, y^*)$, $P_2 = (2x_m - x^*, y^*) \approx (1.2879, 0.9799)$ and $P_3 = (x^*, 2y_m - y^*) \approx (-1.5691, 2.8231)$. Observe that despite the fact that $2x_m - x^*$ and $2y_m - y^*$ are multiple roots of $\mathrm{discr}_y(B)$ and $\mathrm{discr}_x(B)$ respectively, the pair $(2x_m - x^*, 2y_m - y^*)$ is not singular.

Proposition 7 (*Case 4a*)**.** *The biquadratic* $B = 0$ *of (1) contains exactly two distinct singular points* $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *for which* $x_1 \neq x_2$ *and* $y_1 \neq y_2$ *if and only if* $\mu = \mu_{d,x}$, $f_{2XY} = 0$, $f_{2X} \neq 0$ *and* $\mathrm{discr}_x(P_{2X}) \neq 0$.

Proof. Suppose $P_1, P_2 \in B_S$ satisfy the above. Cases 2 and 3 of proposition 2 ensure $\mu = \mu_{d,x} = \mu_{d,y}$, $f_{2X}f_{2XY} = 0$, and x_1, x_2 and y_1, y_2 are the zeros of P_{2X} and P_{2Y} respectively (whose distinctness implies $\mathrm{discr}_x(P_{2X}) \neq 0$ and $\mathrm{discr}_y(P_{2Y}) \neq 0$). We shall prove that $f_{2X} \neq 0$ by contradiction. First, the assumption that $f_{2X} = 0$ ensures $\alpha\epsilon - \beta\delta \neq 0$ else

$\Phi = 0$ by (31). Further, $\alpha = 0$ implies $\mu = \mu_{d,x} = \mu_{a,x}$ (see the proof of proposition 5) and so $y_1 = y_2$ by corollary 1(a), contradicting the distinctness of y_1 and y_2. Similarly, $\alpha \neq 0$ combined with $\alpha\epsilon - \beta\delta \neq 0$ and $\mu = \mu_{d,x}$ yields the contradiction $y_1 = y_2$ by corollary 1(d).

Conversely, suppose $\mu = \mu_{d,x}$, $f_{2XY} = 0$ but $f_{2X} \neq 0$ and $\mathrm{discr}_x(P_{2X}) \neq 0$. Let x_1, x_2 be the distinct zeros of P_{2X}. Then (43), (44), (46) and $P'_{2X} = 0$ of (47) are all valid and combine to ensure (39)–(42), and hence x_1 and x_2 are double roots of $\mathrm{discr}_y(B)$. Proposition 1(b) now ensures that for $y_1 = -\frac{\alpha_1(x_1)}{2\alpha_2(x_1)}$ and $y_2 = -\frac{\alpha_1(x_2)}{2\alpha_2(x_2)}$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ belong to B_S (noting that $\alpha_2(x_i) \neq 0$ else $\mathrm{discr}_y(B)(x_i) = 0 \Rightarrow \alpha_1(x_i) = 0 \Rightarrow f_{2Y} = 0$ by (30) and this would imply $f_{2X} = 0$ by (26)). The distinctness of y_1 and y_2 is due to lemma 6 (which would otherwise establish that $f_{2X} = 0$).

The existence of a third singular point not equal to either P_1 or P_2 would contradict our assumption that $f_{2X} \neq 0$ (via proposition 6). □

An interesting geometric phenomenon related to proposition 7 is illustrated with the aid of the zeros, $x_b^\pm$ and $y_b^\pm$, of α_2 and β_2 respectively. By (6) it is clear that for fixed parameters $K_1, \ldots, K_q$, the lines $x = x_b^\pm$ and $y = y_b^\pm$ represent the vertical and horizontal asymptotes of (1) respectively. Each of the two lines joining the diagonally opposed intersections of these asymptotes has a finite non-zero gradient whose square is

$$M = \frac{\mathrm{discr}_y(\beta_2)}{\mathrm{discr}_x(\alpha_2)}. \tag{98}$$

The following result shows that the line connecting the points P_i of proposition 7 is parallel to one of these diagonal 'asymptote' lines (see figure 6).

Lemma 8. *Suppose P_1 and P_2 are singular points of (1) satisfying the conditions of proposition 7. Then the gradient of the line joining these points is equal to the gradient of one of the diagonal lines joining (x_b^-, y_b^-), (x_b^-, y_b^+), (x_b^+, y_b^-) and (x_b^+, y_b^+).*

Proof. First note that by the proof of proposition 7, $f_{2XY} = 0$ and x_i and y_i of P_1 and P_2 satisfy the identities (39)–(42) and their analogues for $\mathrm{discr}_x(B)/\mathrm{discr}_x(\alpha_2)$ respectively. It is enough to show that for $G = [(y_2 - y_1)/(x_2 - x_1)]^2$, $G - M = 0$ (recalling (98)).

In the case where $C_{x,3} \neq 0$ and $C_{y,3} \neq 0$, we have

$$G = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 = \frac{(y_1 + y_2)^2 - 4y_1y_2}{(x_1 + x_2)^2 - 4x_1x_2} = \frac{(C_{y,3}C_{y,2} - 6C_{y,1})C_{x,3}}{(C_{x,3}C_{x,2} - 6C_{x,1})C_{y,3}}, \tag{99}$$

and the result follows by

$$\begin{aligned} G - M &= \frac{(C_{y,3}C_{y,2} - 6C_{y,1})C_{x,3}}{(C_{x,3}C_{x,2} - 6C_{x,1})C_{y,3}} - \frac{\mathrm{discr}_y(\beta_2)}{\mathrm{discr}_x(\alpha_2)} \\ &= -\frac{12\,\mathrm{discr}_y(\beta_2)\,[\alpha(\beta\xi - \gamma\epsilon) - \gamma(\alpha\epsilon - \beta\delta)]\,\Phi f_{2XY}}{y_m\,\mathrm{discr}_x(\alpha_2)^2\,\mathrm{discr}_x(P_{2X})} = 0. \end{aligned}$$

In the case where $C_{x,3} = 0$ but $C_{y,3} \neq 0$, we have $C_{x,2} \neq 0$, $\mu = \mu'$, $\xi = \xi'$ of case 2 of proposition 2, and

$$\begin{aligned} G - M &= -\frac{(C_{y,3}C_{y,2} - 6C_{y,1})\mathrm{discr}_x(\alpha_2) + 2C_{y,3}C_{x,2}\mathrm{discr}_y(\beta_2)}{2C_{y,3}C_{x,2}\mathrm{discr}_x(\alpha_2)}\bigg|_{(\mu,\xi)=(\xi',\mu'(\xi'))} \\ &= \frac{12\epsilon f_{2XY}|_{\xi=\xi'}}{\delta C_{y,3}C_{x,2}\,\mathrm{discr}_x(\alpha_2)^2\,\mathrm{discr}_y(\beta_2)} = 0. \end{aligned}$$

Similar computations show that $G - M = 0$ in the cases where $C_{y,3} = 0$ but $C_{x,3} \neq 0$ and $C_{x,3} = C_{y,3} = 0$. □

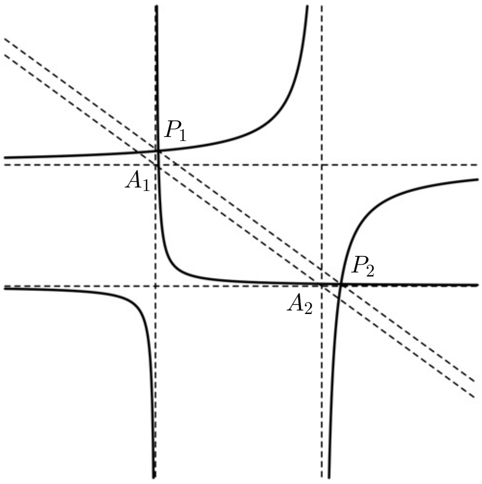


Figure 6. Illustration of case 4a of table 2 (see proposition 7 and lemma 8). When $\mu = \mu_{d,x}$ and $f_{2XY} = 0$ but $f_{2X} \neq 0$ and $\mathrm{discr}_x(P_{2X}) \neq 0$, (1) possesses precisely two singular points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ whose corresponding coordinates lie around the midpoints x_m and y_m (as in (46) and (54)). The gradient of the line connecting P_1 and P_2 (which equals $-\sqrt{M}$ recalling (98)) is identical to that of the line joining A_1 and A_2.

Example 5. The biquadratic

$$B = Kx^2y^2 + ((K-1)t^3 - K^2 + 1)x^2y - xy^2 + x^2 + y^2 + (K+2)x - y + K - t = 0 \tag{100}$$

(which is neither QRT nor McMillan) yields

$$\begin{aligned} f_{2XY} = {} & (4K^3 - 1 - 6K^2 + 4K - K^4)t^{12} + (-14K^4 + 4K^5 - 7K + 3 + 15K^3 - K^2)t^9 \\ & + (-6K^4 - 5 + 18K^5 - 13K^3 - 3K^2 - 6K^6 + 15K)t^6 \\ & + (4K^7 - 12K^2 - 9K^5 + 5 - 10K^6 - 17K + 5K^4 + 34K^3)t^3 \\ & - 1 + 5K + K^5 - 28K^3 - K^8 - 34K^4 + 6K^2 + 2K^7 + 6K^6. \end{aligned} \tag{101}$$

The polynomial $\mathrm{res}_t(f_{2XY}, 4\mathrm{discr}_y(\beta_2)\Phi(\mu - \mu_{d,x}))$ is univariate in K, one of whose roots is $K \approx -1.8144$. Back substitution provides $t \approx -1.0274$ and thus $\mu_{d,x} \approx -0.7870$. Figure 6 makes it clear that the line joining $A_1 = \left(x_b^-, y_b^+\right) \approx (-1.0675, 0.9809)$ and $A_2 = \left(x_b^+, y_b^-\right) \approx (0.5163, -0.5619)$ is parallel to that which joins the two singularities $P_1 \approx (-1.0405, 1.1592)$ and $P_2 \approx (0.6993, -0.5355)$. Clearly $(x_m, y_m) \approx (-0.1706, 0.3118)$ is the midpoint of P_1 and P_2.

Proposition 8 (*Case 5*). *The biquadratic $B = 0$ of (1) contains a singular point $P = (x_1, y_1)$ for which x_1 and y_1 are triple but not quadruple roots of* $\mathrm{discr}_y(B)$ *and* $\mathrm{discr}_x(B)$ *respectively if and only if $\mu = \mu_t$, $f_{3XY} = 0$, $8C_{x,2} - 3C_{x,3}^2 \neq 0$ and $8C_{y,2} - 3C_{y,3}^2 \neq 0$.*

Proof. The necessary direction is a restatement of case 5 of proposition 2 (recalling that $8C_{x,2} - 3C_{x,3}^2 \neq 0$ and $8C_{y,2} - 3C_{y,3}^2 \neq 0$ are a consequence of the triple root conditions on x_1 and y_1 respectively).

For the sufficient direction, let $\mu = \mu_t$, $f_{3XY} = 0$, put $8C_{x,2} - 3C_{x,3}^2 \neq 0$ and $8C_{y,2} - 3C_{y,3}^2 \neq 0$ and define $x_1 = x_t$ (of (66)), $x_2 = -3x_t - C_{x,3}$. It is easily verified that f_{3XY} divides both $F_{x,\mu}(\mu_t)$ and $G_{x,\mu}(\mu_t)$ (of (70) and (73)) and so (68) and (71) are valid.

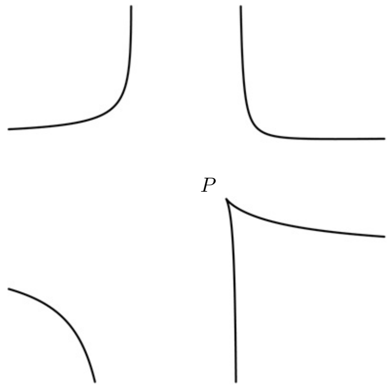


Figure 7. Illustration of case 5 of table 2 (see proposition 8).

Now

$$
\begin{aligned}
&-3x_1 - x_2 - C_{x,3} = 0\\
&3x_1^2 + 3x_1x_2 - C_{x,2} = -\frac{G'_{x,\mu}}{\left(8C_{x,2} - 3C_{x,3}^2\right)^2} = 0\\
&-x_1^3 - 3x_1^2x_2 - C_{x,1} = \frac{\left(C_{x,3}^3 - 16C_{x,1}\right)G'_{x,\mu}}{2\left(8C_{x,2} - 3C_{x,3}^2\right)^3} = 0\\
&x_1^3x_2 - C_{x,0} = x_1^3x_2 - \frac{1}{12}\left(3C_{x,3}C_{x,1} - C_{x,2}^2\right) \text{ (using (68))}\\
&\qquad\qquad\;\; = \frac{SG'_{x,\mu}}{24\left(8C_{x,2} - 3C_{x,3}^2\right)^4} = 0,
\end{aligned}
\tag{102}
$$

where $S = 60C_{x,3}^4C_{x,2} - 9C_{x,3}^6 + 36C_{x,3}^3C_{x,1} - 156C_{x,3}^2C_{x,2}^2 + 48C_{x,3}C_{x,2}C_{x,1} + 128C_{x,2}^3 - 432C_{x,1}^2$, confirm that x_1 is a triple root of $\mathrm{discr}_y(B)$ (by running (60)–(63) in reverse). It is not a quadruple root else $x_t = x_m$ would be a double root of (64), implying $8C_{x,2} - 3C_{x,3}^2 = 0$.

By defining[10] $\hat{y}_1 = -\frac{\alpha_1(x_1)}{2\alpha_2(x_1)}$, we know by the proof of proposition 1(b) that $(x_1, \hat{y}_1) \in B_S$. It follows by (57) that $\hat{y}_1$ is a triple root of $\mathrm{discr}_x(B)$. It is not a quadruple root of this discriminant else $\hat{y}_1 = y_m$ would be a double root of (64)'s correspondent in y, implying $8C_{y,2} - 3C_{y,3}^2 = 0$. □

Example 6. For the biquadratic

$$
\begin{aligned}
B_{\mathrm{QRT}} = {} & ((-3K^2 - 2K + 1)t - 3K + 1)x^2y^2 + ((8K + 21)t - 17K^2 + 41K - 1)x^2y\\
& + ((2K - 1)t + 3K^2 - 4)xy^2 + ((K^2 - 2)t - 27K + 12)x^2\\
& + ((K - 1)t - 3K + 4)y^2 + ((87K + 3)t + 2K^3 - 3K + 5)xy + (Kt - 17)x\\
& + ((-K^2 + 6)t - K^5 + 2)y + K^2 - 2 - t,
\end{aligned}
\tag{103}
$$

[10] The following argument shows that $\alpha_2(x_1) \neq 0$. By the proof of proposition 1(b) we know that $\alpha_2(x_1) = 0 \Rightarrow \frac{\partial B}{\partial y}(x_1, y) = B(x_1, y) = 0$ for any y, and so $f_{2Y} = 0$, $\mu = \mu_{d,y}$ and $x_1 = x^*$ by proposition 4(d), assuming neither α nor $\alpha\epsilon - \beta\delta$ is zero (cases that can be dealt with in a similar manner to the forthcoming). Now if $(x_1, y) \in B_S$, it is clear by applying VE to both sides of (93) that y is a zero of P_{2Y}. The fact that there can be only one such y-solution (recalling (57) and the discussion that follows it) implies that $\mathrm{discr}_y(P_{2Y}) = 0$. Now a contradiction is drawn via the identity $(8C_{y,2} - 3C_{y,3}^2)|_{\mu=\mu_{d,y}} = -4\mathrm{discr}_y(P_{2Y})/(\mathrm{discr}_x(\alpha_2)^2\Psi^2)$.

f_{3XY} consists of 791 terms (degree 16 in t, 50 in K). The polynomial $\mathrm{res}_t\big(f_{3XY}, \big(b_1a_2^2 + b_3a_1^2 - b_2a_1a_2 - b_3a_0a_2\big)(\mu - \mu_t)\big)$ (recalling (74)) is univariate in K (degree 144) and vanishes at $K \approx -0.3054$. Back substitution gives $t \approx 2.5553$, and so also $\mu_t \approx -4.4620$. Note that $8C_{x,2} - 3C_{x,3}^2 \approx -2.0240$ and $8C_{y,2} - 3C_{y,3}^2 \approx -19.3342$ are non-zero and as figure 7 makes clear, the singularity $P = (x_t, y_t) \approx (1.1307, -3.1920)$ is a cusp. In fact, it is easily shown that for $B_y^+ = (-\alpha_1 + \sqrt{\mathrm{discr}_y(B)})/2\alpha_2$, the gradient of the cuspidal tangent, found by calculating $\lim_{h\to 0} \frac{B_y^+(x_1+h) - B_y^+(x_1)}{h}$, is

$$m_c = \frac{\partial}{\partial x}\left(-\frac{\alpha_1}{2\alpha_2}\right)\bigg|_{x=x_t} = \frac{1}{\frac{\partial}{\partial y}\left(-\frac{\beta_1}{2\beta_2}\right)\Big|_{y=y_t}} = -\sqrt{\frac{\beta_2}{\alpha_2}}\bigg|_{(x,y)=(x_t,y_t)} \approx -8.3465.\text{[11]}$$

Proposition 9 (*Case 6*)**.** *The biquadratic $B = 0$ of (1) contains a singular point $P = (x_1, y_1)$ for which x_1 is a quadruple root of* $\mathrm{discr}_y(B)$ *and y_1 is a triple (but not a quadruple) root of* $\mathrm{discr}_x(B)$ *if and only if $\mu = \mu_{d,x}$, $f_{2X} = \mathrm{discr}_x(P_{2X}) = 0$ and $f_{2XY} \neq 0$.*

Proof. The necessary direction is covered by case 6 of proposition 2. For the sufficient direction we put $\mu = \mu_{d,x}$, $f_{2X} = \mathrm{discr}_x(P_{2X}) = 0$, $f_{2XY} \neq 0$ and define $x_1 = x_2 = x_m = -\frac{C_{x,3}}{4}$. It is clear that (43), (44) and

$$\mathrm{discr}_x(P'_{2X}) = \mathrm{discr}_x(P_{2X})|_{\mu=\mu_{d,x}} = C_{x,3}\big(C_{x,3}^3 - 16C_{x,1}\big) = 0$$

all hold (referring to (47) for the latter). These identities readily confirm that (39)–(42) are valid and hence that x_1 is a quadruple root of $\mathrm{discr}_y(B)$.[12] The identity

$$\frac{4}{\mathrm{discr}_y(\beta_2)\Phi^2}\,\mathrm{discr}_x(P_{2X}) + \mathrm{discr}_x(\alpha_2)\,\big(8C_{y,2} - 3C_{y,3}^2\big)\big|_{\mu=\mu_{d,x}} = \frac{48 f_{2XY}}{\mathrm{discr}_x(\alpha_2)\,\mathrm{discr}_y(\beta_2)}$$

establishes that $8C_{y,2} - 3C_{y,3}^2 \neq 0$ and so by footnote 10 we know $\alpha_2(x_1) \neq 0$. Now proposition 1(b) ensures that when $y_1 = -\frac{\alpha_1(x_1)}{2\alpha_2(x_1)}$, $(x_1, y_1) \in B_S$, and so it follows by (57) that y_1 is a triple root of $\mathrm{discr}_x(B)$. It is not a quadruple root of this discriminant, else $y_1 = y_m = -\frac{C_{y,3}}{4}$ and the formulae for $\mathrm{discr}_x(B)$ that correspond to (79) and (80) would give the contradiction $8C_{y,2} - 3C_{y,3}^2 = 0$. □

Proposition 10 is a straightforward consequence of proposition 9 and the symmetry principle noted in remark 6.

Proposition 10 (*Case 7*)**.** *The biquadratic $B = 0$ of (1) contains a singular point $P = (x_1, y_1)$ for which y_1 is a quadruple root of* $\mathrm{discr}_x(B)$ *and x_1 is a triple (but not a quadruple) root of* $\mathrm{discr}_y(B)$ *if and only if $\mu = \mu_{d,y}$, $f_{2Y} = \mathrm{discr}_y(P_{2Y}) = 0$ and $f_{2XY} \neq 0$.*

Proposition 11 (*Case 8*)**.** *The biquadratic $B = 0$ of (1) contains a singular point $P = (x_1, y_1)$ for which x_1 and y_1 are quadruple roots of* $\mathrm{discr}_y(B)$ *and* $\mathrm{discr}_x(B)$ *respectively if and only if $\mu = \mu_{d,x}$ and $f_{2XY} = \mathrm{discr}_x(P_{2X}) = 0$.*

Proof. The necessary direction is covered by case 8 of proposition 2, and for the sufficient direction it is clear via propositions 9 and 10 that $x_1 = x_m$ and $y_1 = y_m$ of $P = (x_1, x_2) \in B_S$ satisfy the required multiplicity conditions. □

[11] The fact that B's tangent, $y = m_c(x - x_t) + y_t \approx -8.3465x + 6.2450$, at (x_t, y_t) intersects the biquadratic in a triple root (namely, $x = x_t$) is what defines this point as a cusp.
[12] This is true regardless of whether $C_{x,3} = 0$.

Acknowledgments

This research was financially supported by an Australian Postgraduate Research Award and by the Australian Research Council.

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